PHYSICAL REVIEW VOLUME 136, NUMBER 1B 12 OCTOBER 1964

Symmetrization Postulate and Its Experimental Foundation

A. M. L. MESSIAH
Département de Physique Nucléaire et Physique du Solide, Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvette (Seine et Oise) France

AND

O. W. GREENBERG*
Department of Physics and Astronomy, University of Maryland, College Park, Maryland

(Received 7 March 1963; revised manuscript received 10 June 1964)

The symmetrization postulate (SP) that states of more than one identical particle are either symmetric or antisymmetric under permutations is studied from the theoretical and experimental points of view. The theoretical analysis is carried out within the framework of particle quantum mechanics; the field-theory approach to identical particles using Bose, Fermi, para-Bose and para-Fermi quantization is not considered in this article. Particles not obeying SP can be accommodated in quantum mechanics, provided some modifications are made in the usual quantum-mechanical formalism. The main modification is to replace the usual ray by a many-dimensional "generalized ray" as the representative of a physical state. The properties of one-body measurements of systems having several identical particles are discussed, and the unobservability in such measurements of interferences between states having different irreducible permutation symmetries is pointed out. The condition of indistinguishability of identical particles is formulated precisely, and is analyzed both for interactions which conserve particle number and for general interactions which do not. For such general interactions, with the additional assumptions of time-reversal invariance and of coherence of the states having given values of charge, baryon number, and lepton number, it is shown that there is an absolute selection rule forbidding transitions between states $\mathfrak{F}^{\times}$ which contain any number of particles of species which obey SP but at most one particle of a species not obeying SP, and states which violate SP. Since only states in $\mathfrak{F}^{\times}$ are now avaiable as initial states of experiments, this selection rule forbids production of SP-violating states in any experiment which is feasible at present. Because of this, presently proposed experimental tests of SP are in fact tests of the quantum-mechanical description of identical particles together with time-reversal invariance and coherence of states in a given superselecting sector. The inclusion of internal variables, such as isospin, for particles violating SP is discussed. A comprehensive discussion is given of direct experimental tests of the SP selection rules. Such tests are more difficult to perform than appears at first sight, because in many cases the indistinguishability of identical particles or the conservation laws already imply the consequences of SP. Criteria for valid tests of SP are given. Several different types of tests are described, with illustrative examples of each. A survey is given of the direct experimental evidence for SP for the various particles. The Fermi character of electrons and positrons and of nucleons is accepted, as is the Bose character of photons. There is good evidence for the Bose nature of pions, especially from the absence of 2π decay of K_2^0. There is no direct evidence for the statistics of K, Λ, Σ, Ξ, or μ. Feasible tests are proposed for the statistics of K and of those hyperons which have an asymmetric decay; but no such tests were found for the other hyperons or for μ.

INTRODUCTION AND RESULTS

WHEN dealing with identical particles in quantum mechanics one usually assumes the *symmetrization postulate* (SP), i.e., states containing several identical elementary particles are, according to the species, either symmetric (bosons) or antisymmetric (fermions). Such a postulate has very important experimental consequences, which can be expressed as a selection rule (SP selection rule).

States which cannot be represented by wave functions of the allowed symmetry type are absolutely forbidden.

This is an extremely strong condition, very much stronger than what is implied by the indistinguishability of identical particles. Consider, for illustration, a system containing two π^+ mesons. To postulate that pions are bosons means that all the states with odd angular momentum are absolutely forbidden, whereas the indistinguishability of the two π^+ only implies that interferences between odd and even waves cannot be observed.

It is widely believed that SP is necessary for treating identical particles in a consistent way. It is also widely believed that it is firmly supported by experiment. In fact, the arguments usually given to insert SP in quantum mechanics are of an *ad hoc* nature, and do not proceed unavoidably from first principles.[1] Also, many experimental facts, which at first sight look like significant tests of the SP selection rule, simply follow from the indistinguishability, e.g., the lack of observable interferences mentioned above.

Historically, SP has played a great role in the understanding of atomic phenomena, mainly as a consistent way of inserting the Pauli principle in the formalism of

* Supported in part by the U. S. Air Force through the Air Force Office of Scientific Research under contract AF 49(638)-24.

[1] The well-known rule that a tightly bound composite system of a fixed number of bosons and fermions is Bose or Fermi according to whether there are, respectively, an even or odd number of fermions present [P. Ehrenfest and J. R. Oppenheimer, Phys. Rev. **37**, 333 (1931)], requires modifications when a pair of the composite particles are close together and interacting. We are not concerned with this effect in the present paper. The general permutation symmetry of concern here is an intrinsic property not derivable from bound states of bosons and fermions.

quantum mechanics. Then, since it was undoubtedly well established for electrons and photons, SP was assumed to apply to all the other elementary particles. This was assumed without too much question, and led to a great simplification in the theory. Granted the fact that it is an extremely good working hypothesis, we believe that both its theoretical justification and the experimental evidences for the corresponding selection rule deserve a careful investigation. Here is the subject of the present paper.

The paper is divided into two parts, each of which is essentially self-contained, the first one about the theoretical formalism, the second about the experimental confirmation.

The first part consists in a general discussion of the problem of identical particles in quantum mechanics, without assuming SP.

In the approach to this problem, one can use either the framework of particle quantum mechanics (QM) or that of field theory (FT). In QM, that is in the usual quantum-mechanical formalism using Fock space, a label is attached to each particle, and it is necessary to introduce permutation operators in order to express the identity of particles. In FT, identical particles are generated by a single field; no such things as permutation operators have to be introduced and the "symmetry properties" of the states are essentially contained in the algebraic relations between field operators. Fields obeying the symmetrization postulate are characterized by bilinear algebraic relations, i.e., commutation relations for Bose fields and anticommutation relations for Fermi fields. More complicated relations lead to fields which are neither Bose nor Fermi. For instance, the simple trilinear relations proposed by Green[2] lead to the so-called parafields, which are in general neither Bose nor Fermi. The QM and FT approaches are known to be equivalent in the case of Bose and Fermi fields. It is still open, however, whether or not the equivalence hold in general. *Here, we stick to the* QM *approach.* The results which are arrived at are valid only for systems which can be consistently described in the framework of QM. We will discuss the FT approach (parastatistics) in a second paper.

We start (Sec. I.1) by formulating precisely the requirement of indistinguishability of identical particles, in terms of invariance of the physical properties of states in the permutation operation. This is the basis of the whole discussion. We first analyze its consequences for systems which conserve the number of identical particles, which is the only case considered in previous treatments (Sec. I.2). Then, we make the same analysis in the general case (Sec. I.3). To conclude this first part (Sec. I.4), we discuss the question of inserting internal variables like isospin and charge conjugation in the definition of identical particles, and give a list of the most relevant SP selection rules.

[2] H. S. Green, Phys. Rev. **90**, 270 (1953).

It turns out that particles not obeying SP can be accommodated in the QM framework without violating any basic principle, pending some modification to the current formulation of quantum mechanics, the main one being to replace the usual ray by a many-dimensional "generalized ray" as the representative of a physical state. Thus, the requirement that identical particles be indistinguishable does not, by itself, imply SP. However, this requirement does impose severe restrictions on the physical observables of the theory and, because of this, also restricts the interactions which the particles can undergo.

For interactions conserving the number of particles, there is a superselection rule which absolutely prohibits transitions between states transforming under inequivalent representations of the permutation group. It follows from this superselection rule that SP can be consistently inserted in the formalism, although it is in no way implied by the axioms of the theory.

The study of the general case leads us to a much stronger result. We find that, under quite broad assumptions, the SP selection rule is verified in all states produced from the initial states which are at present available experimentally, i.e., states in which any particles occurring more than once belong to a species obeying SP. Call $\mathfrak{F}^{\times}$ the space of state vectors belonging to this category. The precise statement of our result is that transitions between vectors in $\mathfrak{F}^{\times}$ and vectors which violate SP are absolutely forbidden, provided that (a) the interactions are time-reversal invariant, and (b) each of the subspaces of $\mathfrak{F}^{\times}$ corresponding to given values of the usual superselecting operators (charge, baryon and lepton number) is fully coherent. Condition (a), time-reversal invariance, seems firmly established by experiment. As for condition (b), it is a rather natural condition to insert in the theory, since it simply means that the algebra of all physical observables is irreducible in each of the subspaces of $\mathfrak{F}^{\times}$ considered. Thus, if a violation of the SP selection rule were found from states belonging to $\mathfrak{F}^{\times}$, one could seriously question the validity of the QM framework itself.

The second part is devoted to a discussion of the possible direct tests of the SP selection rule and to a survey of the present experimental situation. The validity of SP is well established for electrons, nucleons and photons. We are concerned here with all the other elementary particles. We first give (Sec. II.1) the main features of the tests. Then (Sec. II.2), we describe, with numerous illustrative examples, the different types of test which can be used. Finally (Sec. II.3), we make a survey of the present situation for pions, hyperons, kaons and muons.

It turns out that, contrary to expectation, significant tests of the SP selection rules are very hard to obtain. The main reason for this is that there is no observable interference between SP obeying and SP violating transitions. At present, we have a convincing evidence that pions are bosons. For all the other particles considered, the situation is still inconclusive, and the possible tests at hand look quite difficult. The most accessible tests that we find are concerned with the Λ. We also propose a number of tests for kaons, which look feasible. On the contrary in some cases like Σ^- and like muons, the possibility of testing SP looks completely out of reach.

Note added in proof. We have assumed in this article that weak interactions were rigorously PC invariant and that the $K_2^0 \to 2\pi$ decay was rigorously forbidden. Since this work was submitted, J. H. Christensen, J. W. Cronin, V. L. Fitch, and R. Turlay [Phys. Rev. Letters **13**, 138 (1964)], have reported that $K_2^0 \to \pi^+\pi^-$ is observed with a branching ratio $\simeq 2\times 10^{-3}$. They conclude that PC is not rigorously conserved in weak interactions, and that K_2^0 is not pure PC$=-$ but contains a PC$=+$ admixture with amplitude $\simeq 2.3\times 10^{-3}$. An alternative to this conclusion is that PC is rigorously conserved, but pions are not bosons and the K_2^0 undergoes a SP-violating decay with a probability amplitude which turns out to be $\simeq 2\times 10^{-3}$ times the probability amplitude of the SP-obeying $K_1^0 \to \pi^+\pi^-$ decay. Whether weak interactions are actually rigorously or approximately PC invariant should be decided by further experiments (e.g., if K_2^0 is a pure PC$=-$, $K_2^0 \to 2\pi^0$ is rigorously forbidden whether pions are bosons or not). Pending a clarification of the weak interaction situation, our statements about the Bose nature of pions have to be qualified: a small violation of PC invariance in weak interactions looks *a priori* quite surprising, since it is hard to imagine an appealing model of weak interactions with such an *approximate* conservation law; however the alternative explanation stated above looks even more surprizing because, if pions are not bosons, it is even harder to understand why the SP-violating decay is so strongly suppressed. One should also modify slightly our discussion of the $K^0\bar{K}^0$ system: if PC invariance is violated by the stated amount the K_1K_1 and K_2K_2 decay modes still correspond to $C=+$, but the K_1K_2 decay mode corresponds to a state which is mainly $C=-$ with a small admixture of $C=+$ amplitude ($\lesssim 1\%$); even then, the observation of the K decays permits a sufficiently precise determination of the quantum number C for use in our tests of SP for kaons.

I. IDENTICAL PARTICLES IN QM AND THE SYMMETRIZATION POSTULATE (SP)

1. The Indistinguishability of Identical Particles

The discussion of identical particles in QM is usually given for systems with a fixed number of particles under the assumption of SP. Here, we want to deal, more generally, with systems with an arbitrary number of particles, and we will not assume SP. State vectors and observables are defined in the Fock space $\mathfrak{F}$, which is the direct sum of spaces $\mathcal{E}^{(N)}$, each of which is spanned by vectors representing states with a fixed number N_s of particles in each species s:

$$\mathfrak{F}=\sum_{(N)} \mathcal{E}^{(N)} \qquad [(N)=(N_1,N_2,\cdots N_s,\cdots)],$$

$\mathcal{E}^{(N)}$ is a direct product of spaces associated with the QM description of single-particle systems. Call $\mathcal{E}_s$ the space associated with a single particle of species s. We have

$$\mathcal{E}^{(N)}\equiv \mathcal{E}_1{}^{(N_1)}\otimes \mathcal{E}_2{}^{(N_2)}\otimes\cdots\otimes \mathcal{E}_s{}^{(N_s)}\otimes\cdots,$$
$$\mathcal{E}_s{}^{(N_s)}=\bigotimes^{N_s}\mathcal{E}_s.$$

Inside each $\mathcal{E}^{(N)}$, one can define operators describing the permutation of particles belonging to the same species. These permutation operators form a group[3] that we denote by $\mathcal{S}^{(N)}$. Their definition is given in standard text books and will not be repeated here. Any one of these permutations is a mere reshuffling of the labels attached to the particles belonging to the same species. Since these particles are identical, it must not lead to any observable effects. Thus, our basic requirement that identical particles cannot be distinguished is expressed by the following:

Dynamical states represented by vectors which differ only by a permutation of identical particles cannot be distinguished by any observation at any instant of time.

This requirement has consequences both on the properties of physical observables and on the law of evolution of states in the course of time. We first discuss these in the case of systems with a fixed number of particles and show that the standard arguments which lead to the symmetrization postulate in this case imply further assumptions of mere heuristic value and, consequently, can be put to question. Then we discuss the general case and show that, under very broad assumptions, transitions between states obeying the symmetrization postulate and states violating the postulate are absolutely forbidden.

2. QM with a Fixed Number of Particles

In this subsection, we restrict ourselves to one of the subspaces $\mathcal{E}^{(N)}$ defined above. For simplicity, we consider only systems containing one species of particles. This will avoid inessential complications. Thus, $\mathcal{E}^{(N)}$ is the space associated with systems of N particles of the same species.

Permutation Invariance of Physical Observables and of the Evolution Operator

The result of a measurement in a QM system always can be expressed as the expectation value of a suitably

[3] This group is the direct product of the permutation groups of $N_1, N_2, \cdots N_s, \cdots$, objects, respectively.

defined Hermitian operator, A say. The fact that dynamical states represented by $|u\rangle$ and by the permuted vector $P|u\rangle$ cannot be distinguished in a measurement is thus expressed by the equality

$$\langle u|A|u\rangle = \langle u|P^{-1}AP|u\rangle . \tag{1}$$

This must hold for any $|u\rangle$. Applying it to the two superpositions $|u\rangle+\alpha|v\rangle$ and $|u\rangle+i\alpha|v\rangle$ gives[4]

$$\langle u|A|v\rangle = \langle u|P^{-1}AP|u\rangle , \quad u,v\epsilon\mathcal{E}^{(N)}$$

or, equivalently,

$$[P,A]=0 . \tag{2}$$

This must hold for all the possible permutations of the N identical particles. Thus, *all physical observables* (i.e., all observables associated with an actual measurement) *must be permutation-invariant.*

Furthermore, dynamical states represented by $|u\rangle$ and $P|u\rangle$ at time 0 should not exhibit any observable difference at any later time t. The condition implied by this requirement on the evolution operator $U(t)$ is readily obtained by replacing A by $U^\dagger AU$ in the above argument:

$$[P,U^\dagger(t)AU(t)]=0 . \tag{3}$$

$U^\dagger AU$ *must be permutation-invariant*, and this must hold for any physical observable A and for any value of t. Now, since the Hamiltonian of the system is a physical observable, it is permutation-invariant, and one deduces easily from this that $U(t)$ is permutation-invariant too. Thus, the above condition is automatically fulfilled.

Maximal Observation and Generalized Ray

The permutation invariance found here is a rigorous invariance property. It holds for the evolution operator and for *all* physical observables. Contrary to what is implicitly assumed in ordinary QM, the state-vector space $\mathcal{E}^{(N)}$ is not irreducible with respect to the algebra of physical observables. This is precisely the situation which leads to a superselection rule. The discussion here will be slightly more complicated than the usual one because the algebra of the superselecting operators (i.e., the permutation operators) is not Abelian.

The first point to be discussed is the preparation of a dynamical state through physical observations and the extent to which its representative vector can be defined in a preparation. A preparation consists in the performance of simultaneous compatible measurements on the system, with the result that the state vector belongs to one of the common eigensubspaces of the corresponding commuting physical observables. Since these are permutation-invariant, the eigensubspace is also permutation-invariant. In general, it is reducible. Clearly, the most complete preparation is achieved when the eigensubspace is irreducible: no additional commuting physical observable can separate vectors within this eigensubspace. In that case, the preparation will be said to be *maximal.*[5] It gives the maximum amount of information compatible with the indistinguishability of identical particles.

The interesting point to note here is that the irreducible eigensubspace may have dimension greater than 1, in which case the lack of knowledge of the state vector is greater than in the ordinary QM, where the state vector is determined up to a phase or, in more technical language, the QM system is represented by a ray in Hilbert space: it seems natural to call the set of normalized vectors in such an r-dimensional irreducible subspace a "generalized ray" and to say that when the preparation is maximal, the state of a system of identical particles corresponds to a generalized ray, in analogy with the use of the word ray in ordinary QM.

The indeterminate phase factor associated with a state vector in ordinary quantum mechanics does not cause difficulty in the interpretation of the theory because observable results are expressed in terms of absolute values squared of matrix elements from which the indeterminate phase disappears. It is important to realize that the larger indeterminacy with which we are faced here does not cause any difficulty either.

To see this, we have to verify that measurable results on a state associated with a generalized ray do not depend on which state vector in the r-dimensional subspace is chosen to represent the state. Suppose that at time t we perform the measurement associated with the physical observable A on a quantum system which was prepared at time 0 in the ray associated with the irreducible subspace $\mathcal{E}_{\gamma\tau}$ (γ labels the irreducible representation and τ stands for some additional quantum numbers). The result is equal to the expectation value of $U^\dagger(t)AU(t)$ for the initial state vector. Now, since $U^\dagger AU$ is permutation-invariant and since $\mathcal{E}_{\gamma\tau}$ is irreducible, Schur's lemma implies that the expectation value is the same for all normalized vectors in $\mathcal{E}_{\gamma\tau}$:

$$\langle u|U^\dagger(t)AU(t)|u\rangle = \langle v|U^\dagger(t)AU(t)|v\rangle ; \qquad |u\rangle,|v\rangle\epsilon\mathcal{E}_{\gamma\tau} . \tag{4}$$

One-Body Measurements

One-body measurements deserve special attention. By a one-body measurement, we mean a measurement on each particle taken separately. Consider, for example,

[4] The proof given here has to be amended if $\mathcal{E}^{(N)}$ is split by superselection rules. Strictly speaking, our argument proves Eq. (2) if $|u\rangle$ and $|v\rangle$ belong to the same superselecting sector (assumed to be fully coherent), since it assumes that any linear combination of these two vectors represents a dynamical state. If $|u\rangle$ and $|v\rangle$ belong to different superselecting sectors, the two matrix elements below vanish and Eq. (2) still holds: the left-hand side vanishes because A is a physical observable, the right-hand side because in addition the action of P, which does not change the physical properties of states, *a fortiori* leaves the superselecting sectors invariant. The same amendments apply as well to the proofs of relation (3) and relation (14) below.

[5] In the same line of argument, we have to replace the usual notion of complete set by the notion of maximal set of commuting physical observables, by which we mean a set, whose eigenvalues define subspaces irreducible with respect to permutation.

a two-electron system. The determination of their respective momenta, or of their respective spins, or of both momenta and spins, are typical one-body measurements. On the contrary, such quantities as the total spin or the relative angular momentum, which imply some correlation between the two electrons, do not correspond to one-body measurements.

Strictly speaking, all the experiments in elementary-particle physics are collision experiments and always consist of a set of one-body measurements, since the observations are performed on individual, widely separated, noninteracting particles. Thus, quantities which imply a correlation between the particles cannot be measured directly. Information about these quantities can be obtained only through the study of correlations between several sets of one-body measurements (e.g., angular correlation experiments), and their determination through such studies always requires some assumption about the dynamical properties of the observed system. This will be illustrated in the examples given in Sec. II. For the moment, we simply want to stress the importance of one-body measurements.

In general, *a set of one-body measurements, no matter how complete, is not maximal.*

The most complete set of one-body measurements is a set in which the dynamical state of each particle is completely determined (energy-momentum 4-vector, helicity, and possible internal quantum numbers such as various charges). Since this gives the list of quantum numbers of each particle separately, but does not give any information about the permutation symmetry of the many-particle states, such a measurement depends only on the *sum* of projection operators into each of the irreducible representations which may occur. Clearly, the number of these is in general greater than one, hence the measurement is not maximal.

To see this in more detail, we take first the simple example of a system containing two identical particles. The measurement will consist in taking the probability that we find one particle in quantum state λ_1 and the other in quantum state λ_2. According to the standard rules of quantum mechanics, it is given by the expectation value of the projector $\Lambda(\lambda_1,\lambda_2)$ onto the subspace corresponding to this set of quantum numbers. In general, this subspace will be two-dimensional, and a convenient orthonormal basis in it will be given by the normalized symmetric and antisymmetric state vectors. We denote these by $|s\rangle$ and $|a\rangle$, respectively. The projector can be written

$$\Lambda(\lambda_1,\lambda_2)=|s\rangle\langle s|+|a\rangle\langle a|\,.$$

If we denote by $|\Psi\rangle$ the state vector of the system, the desired probability is

$$w(\lambda_1,\lambda_2)=\langle\Psi|\Lambda(\lambda_1,\lambda_2)|\Psi\rangle$$
$$=|\langle\Psi|s\rangle|^2+|\langle\Psi|a\rangle|^2. \qquad (5)$$

We draw the reader's attention to two features of the observed quantity w, which follow from the fact that the particles are identical and that $\Lambda(\lambda_1,\lambda_2)$ is permutation-invariant: (i) symmetry in λ_1 and λ_2; (ii) absence of interference terms between symmetric and antisymmetric states.

Consider now the general case, where there are N identical particles in single-particle states $\lambda_1, \lambda_2, \cdots, \lambda_N$. The observable to be associated with the measurement is the projector $\Lambda(\lambda_1,\lambda_2,\cdots,\lambda_N)$ onto the subspace $\mathcal{E}^{[\lambda]}$ spanned by all vectors deduced by permutation from the product vector $|\lambda_1\rangle|\lambda_2\rangle\cdots|\lambda_N\rangle$. $\mathcal{E}^{[\lambda]}$ is, of course, permutation-invariant. Except in the very particular case when $\lambda_1=\lambda_2=\cdots=\lambda_N$, it is reducible. In general, all the λ's are different and $\mathcal{E}^{[\lambda]}$ is associated with the regular representation of the permutation group. As is well known, the decomposition of the regular representation contains each irreducible representation of the group *a number of times equal to its degree*. In particular, the representations which occur only once are the one-dimensional representations, i.e., the symmetrical and the antisymmetrical.

It is convenient to decompose $\mathcal{E}^{[\lambda]}$ into irreducible components $\mathcal{E}_{(\gamma\tau)}{}^{[\lambda]}$. γ denotes the representation to within an equivalence and τ is an additional quantum number for representations which occur more than once. For a given representation $(\gamma\tau)$, we pick an orthonormal basis $|\gamma\mu\tau\rangle$ ($\mu=1, 2, \cdots, r$, where r is the degree of the representation γ). Using this basis we get

$$\Lambda(\lambda_1,\cdots\lambda_N)=\sum_{\gamma\tau}\Lambda_{(\gamma\tau)}{}^{[\lambda]}\,, \qquad (6)$$

$$\Lambda_{(\gamma\tau)}{}^{[\lambda]}=\sum_{\mu}|\gamma\mu\tau\rangle\langle\gamma\mu\tau|\,. \qquad (7)$$

The probability that a one-body measurement on a system in state $|\Psi\rangle$ yields single-particle quantum numbers $\lambda_1, \cdots, \lambda_N$ is

$$w(\lambda_1,\cdots,\lambda_N)=\langle\Psi|\Lambda(\lambda_1,\cdots,\lambda_N)|\Psi\rangle \qquad (8)$$
$$=\sum_{\gamma\tau}\langle\Psi|\Lambda_{(\gamma\tau)}{}^{[\lambda]}|\Psi\rangle$$
$$=\sum_{\gamma\tau\mu}|\langle\Psi|\gamma\mu\tau\rangle|^2. \qquad (9)$$

Again we note that this observed quantity has the properties (i) symmetry in the λ's (ii) absence of interference terms between vectors in inequivalent representations.

For use in Sec. II, we state corresponding results for the T matrix. We first define T-matrix elements from an initial state $|\varphi\rangle$ to a final N-particle state $|\gamma\mu\tau\rangle$ by

$$T_{\gamma\mu\tau}=\langle\gamma\mu\tau|T|\varphi\rangle\,.$$

Up to phase-space factors, the cross section is given by

$$\sigma(\lambda_1\cdots\lambda_N)\propto\sum_{\gamma\tau\mu}|T_{\gamma\mu\tau}|^2. \qquad (10)$$

SP and Superselection Rule between Symmetry Types

Vectors which transform, to within an equivalence according to a given irreducible representation of $\mathcal{S}^{(N)}$, will be said to have a given symmetry type. We define the degree of a symmetry type as the degree of the associated irreducible representation. We use the notation $\mathcal{S}_\gamma{}^{(N)}$ to denote a symmetry type, and $\mathcal{E}_\gamma{}^{(N)}$ to denote the subspace spanned by vectors of the same symmetry type, γ labeling here the relevant irreducible representation. Particular subspaces of interest are those associated with the types of degree 1, namely the spaces $\mathcal{E}_S{}^{(N)}$ and $\mathcal{E}_A{}^{(N)}$ spanned by the symmetric and antisymmetric vectors, respectively. $\mathcal{E}^{(N)}$ is the direct sum of all these subspaces:

$$\mathcal{E}^{(N)}=\sum_\gamma \mathcal{E}_\gamma{}^{(N)}. \tag{11}$$

Maximal observations, as defined above, lead to state vectors of a definite symmetry type. Therefore, it is quite natural to advance the postulate that only one type occurs in nature, otherwise stated that all representative vectors belong to one and the same component $\mathcal{E}_\gamma{}^{(N)}$ of $\mathcal{E}^{(N)}$. SP is a somewhat restricted form of this postulate, in that it is further assumed that the allowed component has to be either $\mathcal{E}_S{}^{(N)}$ or $\mathcal{E}_A{}^{(N)}$.

That the postulate can be consistently inserted in the QM framework can be seen by proving that *a superselection rule operates between vectors of different symmetry types.*

The proof follows essentially the same line as the one given in the paragraph on maximal observations.

Applying Schur's Lemma, we note first that due to the permutation invariance of $U(t)$, vectors in an irreducible representation remain in an equivalent representation in the course of time. Therefore vectors of a definite symmetry type keep this symmetry type in the course of time. Secondly, since physical observables are permutation invariant, they cannot have nonvanishing matrix elements between subspaces of different symmetry type.

Conditions under Which Only Bose and Fermi Particles Can Occur

The literature of physics contains a number of discussions which purport to show that only Bose and Fermi particles can occur. We want to point out that the arguments, when formulated correctly, always imply an additional assumption.

Two rather independent arguments are usually given.

Expressed in its simplest way, the first argument is based on the requirement that

$$P|u\rangle=c|u\rangle,\quad |c|=1, \tag{12}$$

i.e., that permuting the particles in a state vector changes it only by a numerical phase factor. This requirement is precisely the requirement that the representative vectors transform as a one-dimensional representation of the permutation group, which in turn is the same as requiring that the representation be symmetric or antisymmetric.

Relation (12) complies with our basic requirement of indistinguishability, but it is stronger. It cannot be deduced from this requirement alone. One has to assume in addition that dynamical states which cannot be distinguished by any observation are represented by the same vector to within a phase factor.

Proofs of the postulate have been produced recently[6] which are based on essentially the same argument. It is assumed as a starting point that there "must exist a complete set of commuting observables," which is, expressed in a more learned way, the same assumption about representative vectors as the one given above.

This assumption is usually inserted in QM for heuristic purposes, and can be relaxed without contradicting any basic principle of the theory.[5]

The second argument starts from the consideration of one-body measurements. It is required that a complete set of one-body measurements, like the one described in the above paragraph, be maximal. Clearly, the only way of achieving this result is to postulate that state vectors must have a definite symmetry type *taken among those which occur no more than once in* $\mathcal{E}^{[\lambda]}$, that is, the symmetrical and the antisymmetrical. Hence, the symmetrization postulate.

This requirement about one-body observations is very reasonable. It means that a complete knowledge of the respective dynamical states of the particles taken separately entails a complete knowledge of the system as a whole. However, one could get along without it. Then, states prepared through one-body observations would have to be described by a density matrix. Some practical rule would be needed in order to define this density matrix, but no principle is opposed to this possibility.

In summary, for systems with a fixed number of particles, there is a superselection rule between symmetry types which permits one to insert SP in the quantum theory in a consistent way. However the postulate does not appear as a necessary feature of the QM description of nature. Whether it is followed by nature or not has to be decided by experiment.

3. QM with a Variable Number of Particles

We consider now the consequences of the basic requirement of indistinguishability in the general case, when the number of particles is no longer a fixed quantity.

Some care has to be taken because the permutations are not operations defined in the whole Fock space $\mathfrak{F}$ but in each component $\mathcal{E}^{(N)}$ of $\mathfrak{F}$, for which the number

[6] J. M. Jauch, Helv. Phys. Acta **33**, 711 (1960); J. M. Jauch and B. Misra, *ibid.* **34**, 699 (1961); A. Galindo, A. Morales, and R. Nuñez-Lagos, J. Math. Phys. **3**, 324 (1962); D. Pandres, *ibid.* **3**, 305 (1962).

of particles is fixed. In the same way, observations where the indistinguishability comes into play are observations on a fixed number of particles. Therefore, the physical observables which we consider in the present discussion are observables defined in a given subspace $\mathcal{E}^{(N)}$.

The superscripts (M), (N), $\cdots$ will be used to denote objects relating to the subspaces $\mathcal{E}^{(M)}$, $\mathcal{E}^{(N)}$, $\cdots$, respectively. We call $\Lambda^{(M)}$ the projector on $\mathcal{E}^{(M)}$, $A^{(M)}$ a physical observable in $\mathcal{E}^{(M)}$. Note that

$$A^{(M)}=\Lambda^{(M)}A^{(M)}\Lambda^{(M)}.$$

As a straightforward extension of relation (2), we have the result that $A^{(M)}$ is invariant in any permutation $P^{(M)}$ of identical particles in $\mathcal{E}^{(M)}$.

$$[P^{(M)},A^{(M)}]=0. \tag{13}$$

Similarly, the permutation invariance of $U^\dagger A U$ is generalized to the following permutation invariance property:

$$[P^{(N)},\Lambda^{(N)}U^\dagger(t)\Lambda^{(M)}A^{(M)}\Lambda^{(M)}U(t)\Lambda^{(N)}]=0. \tag{14}$$

This should hold for all values of t, for all possible subspaces $\mathcal{E}^{(M)}$, $\mathcal{E}^{(N)}$, for all permutations in $\mathcal{E}^{(N)}$, and for all physical observables in $\mathcal{E}^{(M)}$.

Relation (14), of which relation (13) is a particular case, expresses the indistinguishability of identical particles. The discussion can be pursued along the same line as before, with the added complication that $U(t)$ does not conserve the number of particles any longer, i.e., it has nonvanishing matrix elements between the components of Fock space. The permutation invariance of $U(t)$ is lost, it is even meaningless, and the results which followed from this simplifying feature will have to be amended accordingly.

All the previous comments and statements about maximal observations and generalized rays remain valid. The consistency of the notion of maximal observation lies directly on property (14), from which the generalized version of (4) is easily deduced. All the statements about one-body measurements also remain valid. It is worth recalling, that, in the absence of a postulate restricting the symmetry types, one-body experiments, no matter how complete they are, can never be maximal.

On the other hand, the proof of the superselection rule between symmetry types, which was based on the permutation invariance of $U(t)$, breaks down. Clearly, however, relation (14) imposes on $U(t)$ very severe limitations that we now proceed to investigate. The principal aim of this investigation is to see whether the SP selection rules are mere consequences of relations (14). The answer to this question turns out to be in the negative. But the additional assumptions which are needed to derive the SP selection rules are not very stringent.

First, we prove the following selection rule:

Transitions from a given symmetry type to symmetry types of smaller degree are forbidden.

Consider the transition from a state in the irreducible subspace $\mathcal{E}_{\gamma\tau}^{(N)}$ into a state in the irreducible subspace $\mathcal{E}_{\beta\sigma}^{(M)}$. We call $\Lambda_{\gamma\tau}^{(N)}$, $\Lambda_{\beta\sigma}^{(M)}$ the projectors onto these subspaces, and r_γ, r_β their respective dimensions. We want to show that

$$Q=\Lambda_{\beta\sigma}^{(M)}U(t)\Lambda_{\gamma\tau}^{(N)}=0 \quad \text{if} \quad r_\beta<r_\gamma. \tag{15}$$

From relation (14) taken with $\Lambda_{\beta\sigma}^{(M)}$ for the observable $A^{(M)}$, it follows that $Q^\dagger Q$ is invariant under all permutations $P^{(N)}$ within the irreducible subspace $\mathcal{E}_{\gamma\tau}^{(N)}$, hence

$$Q^\dagger Q=c\Lambda_{\gamma\tau}^{(N)},$$

where c is the transition probability under consideration. Thus, if $c\neq 0$, the range of $Q^\dagger Q$ has necessarily dimension r_γ. This cannot happen if $r_\beta<r_\gamma$, since the dimension of the range of $Q^\dagger Q$, like that of Q, is at most equal to r_β in this case. Then $Q^\dagger Q=0$, hence the result (15). Q.E.D.

In order to proceed further, we obviously have to make specific assumptions about $U(t)$ or about the properties of Fock space.

We assume from now on that $U(t)$ is time-reversal invariant. Then the above selection rule also applies to the time-reversed transitions, which means that transitions to symmetry types of greater degree are also forbidden. Combining the two results, we conclude that:

If the evolution operator is time-reversal invariant, the degree of the symmetry types is conserved in all transitions.[7]

In all present experiments, the initial states have at most one particle of a species, whose permutation character might be questioned (e.g., π, K, Λ, Σ, $\cdots$), and have more than one particle only for species known to be Bose or Fermi, such as nucleons, electrons, or photons. We call $\mathfrak{F}^\times$ the subspace of all such states. Since only one-dimensional representations of the permutation groups occur in $\mathfrak{F}^\times$, all its states have symmetry types of degree 1. Thus, if the above selection rule applies, symmetry types of greater degree are not accessible in present experiments, i.e., in each $\mathcal{E}^{(N)}$,

[7] As can be seen easily, this absolute conservation law generates a superselection rule. The assumption of time-reversal invariance leads to the following even stronger result which we state here without proof. Given a $\mathcal{S}^{(N)}$-irreducible subspace of $\mathcal{E}^{(N)}$, $\mathcal{E}_{\gamma\tau}^{(N)}$ say, the only nonvanishing transitions from its vectors into $\mathcal{E}^{(M)}$ are those to vectors, which have nonvanishing components in a certain $\mathcal{S}^{(M)}$-irreducible subspace of $\mathcal{E}^{(M)}$, $\mathcal{E}_{\beta\tau}^{(M)}$ say (we, of course, have $r_\gamma=r_\beta$). This result can be expressed by

$$\Lambda^{(M)}U(t)\Lambda_{\gamma\tau}^{(N)}=\Lambda_{\beta\sigma}^{(M)}U(t)\Lambda_{\gamma\tau}^{(N)}.$$

One also has, from time-reversal invariance,

$$\Lambda^{(N)}U(t)\Lambda_{\beta\sigma}^{(M)}=\Lambda_{\gamma\tau}^{(N)}U(t)\Lambda_{\beta\sigma}^{(M)}.$$

Applying this result to the case $M=N$, one easily concludes that transitions between different symmetry types in $\mathcal{E}^{(N)}$ are forbidden, even when $U(t)$ does not conserve the number of particles.

the symmetrical and antisymmetrical states are the only ones, the production of which is not forbidden by this rule.

Let us now investigate more closely the transitions from $\mathfrak{F}^\times$. The interesting property of $\mathfrak{F}^\times$ is that maximal observations define a state vector in a unique way, apart from the usual arbitrariness in phase. The converse is certainly not true, i.e., to each vector in $\mathfrak{F}^\times$ there does not necessarily correspond an observable dynamical state of the system, since $\mathfrak{F}^\times$ is decomposed into incoherent sectors by the superselection rules associated with the conservation of the charge Q, the baryon number B and the lepton number L. We denote these sectors by $\mathfrak{F}^\times{}_{QBL}$. Each $\mathfrak{F}^\times{}_{QBL}$ may reasonably be assumed to be "fully coherent." By this, we mean that each vector in $\mathfrak{F}^\times{}_{QBL}$ does correspond to an observable dynamical state. If these conditions are met, deviations from SP are impossible to put into evidence in practice, as a consequence of the following theorem.

Theorem: If the laws of motion are time-reversal invariant,[7a] *and if the superselecting sectors $\mathfrak{F}^\times{}_{QBL}$ of $\mathfrak{F}^\times$ are fully coherent, transitions between states in $\mathfrak{F}^\times$ and states in which some of the identical particles have not the symmetry type required by SP, are absolutely forbidden.*

To illustrate this, consider $di-\pi^+$ states produced in a reaction such that $\pi^+ p \to n\pi^+\pi^+$. The theorem states that, *even if the π^+ are not bosons*, these states cannot be a superposition of symmetrical and antisymmetrical states; either all states produced in this way are purely symmetrical, or they all are purely antisymmetrical, as would be required by SP (the latter possibility is ruled out by experimental evidence). The same result obtains for $di-\pi^+$ produced in the reaction $pp \to nn\pi^+\pi^+$, since protons are known to be fermions.

The proof goes as follows. To start with, we suppose that $\mathfrak{F}^\times$ is fully coherent. The modifications required by the occurrence of superselection rules will be given at the end of the proof.

We focus on a particular species s, whose symmetrical character is put to question, and consider first the transitions between states in $\mathfrak{F}^\times$ and states in a given $\mathcal{E}^{(N)}$ for which $N_s > 1$. The permutations considered all along refer exclusively to species s.

Assume that there is a state $|u\rangle \epsilon \mathcal{E}^{(N)}$ which has a nonvanishing transition probability w at time t to a given state $|\chi\rangle \epsilon \mathfrak{F}^\times$, i.e.,

$$w = |\langle \chi | U(t) | u\rangle|^2 \neq 0.$$

w can be rewritten as the expectation value over $|u\rangle$ of the operator $|v\rangle\langle v|$, with the notation

$$|v\rangle \equiv \Lambda^{(N)} U^\dagger(t) |\chi\rangle, \quad |v> \epsilon \mathcal{E}^{(N)} \quad (\|v\| \neq 0).$$

$|v\rangle\langle v|$ is a multiple of a projector on a one-dimensional space, the space of vectors proportional to $|v\rangle$. According to (14), it is permutation-invariant. Therefore $|v\rangle$ must belong to one of the two possible one-dimensional representations of the permutation group in $\mathcal{E}^{(N)}$. Assume for concreteness that it is the symmetrical one. Then, only vectors which have a nonvanishing component in $\mathcal{E}_S{}^{(N)}$ may have a nonvanishing transition probability to state $|\chi\rangle$.

Next we prove that if transitions to $|\chi\rangle$ are from $\mathcal{E}_S{}^{(N)}$ rather than $\mathcal{E}_A{}^{(N)}$, transitions to any other vectors of $\mathfrak{F}^\times$ from $\mathcal{E}_A{}^{(N)}$ are forbidden. Assume that transitions from $\mathcal{E}_A{}^{(N)}$ to a given vector $|\chi'\rangle \epsilon \mathfrak{F}^\times$ are allowed; the relating transition probabilities are given by the expectation values of $|v'\rangle\langle v'|$, with

$$|v'\rangle \equiv \Lambda^{(N)} U^\dagger(t) |\chi'\rangle, \quad |v'\rangle \epsilon \mathcal{E}_A{}^{(N)}, \quad \|v'\| \neq 0.$$

Now, *since $\mathfrak{F}^\times$ is fully coherent*, there is a state associated with the linear combination

$$|\Psi\rangle = \lambda|\chi\rangle + \mu|\chi'\rangle, \quad (\lambda, \mu \neq 0)$$

and the transition probability to this state is given by the expectation value of $|z\rangle\langle z|$, with

$$|z\rangle \equiv \lambda|v\rangle + \mu|v'\rangle.$$

Obviously $|z\rangle\langle z|$ is not permutation invariant, contrary to what is required by the indistinguishability of identical particles.

In conclusion, only vectors which have components in $\mathcal{E}_S{}^{(N)}$ (or only those with components in $\mathcal{E}_A{}^{(N)}$) can perform transitions to $\mathfrak{F}^\times$. Because of the time-reversal invariance, the selection rule also holds for the time-reversed transitions, that is, transitions from $\mathfrak{F}^\times$ to $\mathcal{E}^{(N)}$ lead only to vectors in $\mathcal{E}_S{}^{(N)}$ (or in $\mathcal{E}_A{}^{(N)}$).

Next, we show that the allowed symmetry type of a given species s is the same for all values of the number N_s of particles in this species. Assume for concreteness that it is symmetric for a given value of N_s. Transitions from a state in $\mathfrak{F}^\times$ to a state with N_s particles in species s lead necessarily to symmetric states. Now, the transition probability must not change, if we add to the initial and final states one particle s sufficiently far away so that it does not interact with any other particle in the course of the reaction. This means that the symmetry of the new final state (N_s+1 particles s) must be such that it is completely symmetric in the permutations of N_s particles. This condition excludes the antisymmetric type.

Finally, we have to extend the result to the case when $\mathfrak{F}^\times$ is no longer fully coherent but is split into superselecting sectors $\mathfrak{F}^\times{}_{QBL}$ which are fully coherent. Then, the whole Fock space $\mathfrak{F}$ is also split into superselecting sectors which we denote by $\mathfrak{F}_{QBL}$. Clearly, the demonstration above can be carried through within each $\mathfrak{F}_{QBL}$ separately. Furthermore, the allowed symmetry type of a given species s is the same for all values of Q, B, and L, since these numbers can be changed in an arbitrary way by adding to the initial and final states of a reaction a suitable number of widely sepa-

[7a] *Note added in proof.* This theorem remains valid if ICP invariance is substituted for time-reversal invariance.

rated noninteracting fermions or bosons[8] without changing the transition probability. Q.E.D.

We repeat that, in practice,[9] all states in elementary-particle physics are produced in collisions from initial states belonging to $\mathfrak{F}^\times$. Consequently, these states should have the symmetry type required by SP and no deviations from the postulate should be observable in present experiments if all the conditions on which the above theorem is based are met.[10] These conditions are:

(i) validity of the QM description of identical particles within a Fock space.

(ii) time-reversal invariance of the laws of motion.

(iii) full coherence of each superselecting sector $\mathfrak{F}^\times_{QBL}$.

We feel that experimental tests of such a drastic selection rule as the SP selection rule are needed anyway, but it is good to realize that these experiments in fact test the set of conditions stated above.

4. Inclusion of Internal Variables and Selection Rules

In the QM description, we have some leeway in the definition of the species of identical particles. For instance, neutrons and protons could be treated either as two different species of particles, or as two different states of the same species with an internal degree of freedom. As is well-known,[11] the two treatments are rigorously equivalent when the symmetrization postulate is made. However, this is no longer true when the symmetry types are not restricted to that extent.

Take, for example, the case of two pions of different charge, a π^+ and a π^0 say, with momenta $\mathbf{k}'$ and $\mathbf{k}''$, respectively. If π^+ and π^0 are treated as different particles, this defines just one state vector to within a phase. If they are treated as identical particles with an internal charge variable, this defines two linearly independent state vectors, $|\mathbf{k}'+, \mathbf{k}''0\rangle$ and $|\mathbf{k}''0, \mathbf{k}'+\rangle$, or any normalized linear combination of these; then, assuming that pions are bosons means that only the symmetrical combination is allowed, and we again find just one state vector.

Quite generally, consider r similar species of identical particles. By similar, we mean here two particles whose dynamical variables and states are unitary equivalent (e.g., n and p, e and n, π^+ and K^-, $\cdots$; in practice, two particles with the same spin). Two similar particles are not necessarily identical, but their permutations can be defined in a consistent way. We consider dynamical states where we have N_s particles in species s ($s=1, 2, \cdots, r$) in the individual states $\lambda_1^{(s)}, \lambda_2^{(s)}, \cdots, \lambda_{N_s}^{(s)}$, respectively. If we do not put any restriction on the symmetry types, the number of thereby defined linearly independent state vectors is in general $\prod_{s=1}^r N_s!$. This number goes up to $(\sum_s N_s)!$, when we treat all particles as belonging to a single species with an internal degree of freedom, which can take r distinct values. On the other hand, if the symmetrization postulate is applied, there is just one state vector thereby defined in both treatments.

It is therefore quite important to decide what internal variables should enter the definition of each species of particle, if one wants to question the SP and submit it to experimental tests. In practice, the inclusion of internal variables is unavoidable when these variables have to be used to express some invariance properties of the interactions. Two invariance laws are of interest to us here, the isospin invariance and the charge conjugation invariance.

Isospin

Since the strong interactions conserve isospin, isospin variables must enter the definition of all strongly interacting particles. Thus, the π^+, π^0, π^- must be treated as the three charge states of a single species of particle, the pion, and the corresponding charge variables must be permuted together with the space-time variables when one performs a permutation on a multipion state.

One can always write the state vectors as a sum of terms, each of which corresponds to a definite value of the total isospin. This decomposition proves quite useful when, as is often the case, the charge wave functions of given total isospin have a definite symmetry type.

We illustrate this procedure in the simple case of a two-pion system. Let k_i and q_i ($i=1, 2$) be the momenta and charges of the two pions. Then the irreducible representations of the permutation P_{12},

$$(k_1q_1,k_2q_2) \longrightarrow (k_2q_2,k_1q_1),$$

are the symmetric states Ψ_s which have the usual Bose symmetry

$$P_{12}\Psi_s(k_1q_1,k_2q_2)=\Psi_s(k_1q_1,k_2q_2)$$

[8] We know that electrons and nucleons are fermions. Any change in QBL can be achieved by adding a suitable number of these particles. Thus, increasing Q by 1 without changing B and L is obtained by adding $\bar{n}p$, a similar increase for B is obtained by adding n, the same for L by adding $e\bar{n}p$.

[9] One can, of course, imagine initial states outside $\mathfrak{F}^\times$, for example the one realized by two *independently produced* beams of kaons, or of muons, etc. Thus a K^+K^+ scattering experiment with clashing beams would fall outside the domain of application of the general theorem above. Apparently, it also falls outside the present experimental possibilities.

[10] This conclusion and the point of view from which it was derived are different from the conclusions and approach of the parafield theory. In parafield theory, which is a particular type of second quantized field theory, the requirement that Hamiltonian density be a local observable (in the sense of spacelike commutativity) leads to certain absolute selection rules, the most important of which is the rule that no para particle can decay entirely into ordinary particles. This rule, together with a $\mu\bar{\mu}$ photoproduction experiment, leads to the conclusion that no presently known particle can be para. We will discuss the selection rules for para particles in a forthcoming paper. The rules (called "conservation of statistics" rules for para particles) stated by S. Kamefuchi and J. Strathdee [Nucl. Phys. 42, 166 (1963)] are incorrect.

[11] A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1962), English ed., Vol. II, Chap. 14.

and the antisymmetric states Ψ_a

$$P_{12}\Psi_a(k_1q_1,k_2q_2)=-\Psi_a(k_1q_1,k_2q_2).$$

For this simple case, only one-dimensional representations occur. Each type of state can be separated into states of given total isospin I and given value of its z component $M=q_1+q_2$. Using the fact that for two particles even I (odd I) isospin wave functions are symmetric (antisymmetric), we obtain

$$\Psi_{s,a}(k_1q_1,k_2q_2)=\phi^{(0)}(q_1q_2)\psi_{s,a}{}^{(0)}(k_1k_2)+\sum_{M=-1}^{1}\phi^{(1M)}(q_1q_2)\psi_{a,s}{}^{(1M)}(k_1k_2)+\sum_{M=-2}^{2}\phi^{(2M)}(q_1q_2)\psi_{s,a}{}^{(2M)}(k_1k_2). \quad (16)$$

Here $\phi^{(1M)}$ are the usual isospin wave functions. The (unnormalized) momentum-space wave functions ψ are symmetric or antisymmetric as indicated by their subscript. If we assume that pions are bosons, only one of these two possible symmetry types is allowed for the ψ according to the well-known rule

$$(\text{Bose } \pi\pi) \quad (-)^I=(-)^J, \quad (17)$$

and the forbidden ψ vanish identically. Here J is the relative angular-momentum quantum number. To write this we have used the fact that symmetric orbital wave functions contain only even waves, and antisymmetric ones only odd waves. Relation (17) is characteristic of Bose pions and could be used for tests of SP for pions.

Charge Conjugation

Since strong and electromagnetic interactions are invariant under charge conjugation (and very probably all interactions are invariant under PC) we are led to treat a particle and its antiparticle as belonging to the same species.

Systems of interest here are particle-antiparticle systems, since they can be chosen to be eigenstates of the charge conjugation operator C which, in this particular case, is nothing but the permutation operator on the relevant internal variables. Typical systems of this kind are $p\bar{p}$, K^+K^-, and $K^0\bar{K}^0$ systems. The internal variable is the quantum number distinguishing particle from antiparticle, e.g., baryon number for $p\bar{p}$ and strangeness for $K\bar{K}$. We denote it by ϵ, $\epsilon=+1$ corresponding to the particle and $\epsilon=-1$ to the antiparticle. The operation of C replaces ϵ by its opposite.

For illustration, let us discuss in some detail the $K^0\bar{K}^0$ system. It appears here as a particular di-kaon system, whose wave function $\Psi(k_1\epsilon_1,k_2\epsilon_2)$ vanishes when $\epsilon_1+\epsilon_2\neq 0$. Clearly, the C operation is equivalent to exchanging ϵ_1 and ϵ_2 in this particular case.

Now, exactly as we did with isospin on the 2π system, we can separate in the wave function the terms corresponding to the two possible eigenvalues of C. Thus, for a wave function of definite symmetry type we obtain

$$\Psi_{s,a}(k_1\epsilon_1,k_2\epsilon_2)=\chi^{(+)}(\epsilon_1\epsilon_2)\psi_{s,a}{}^{(+)}(k_1,k_2)+\chi^{(-)}(\epsilon_1\epsilon_2)\psi_{a,s}{}^{(-)}(k_1,k_2). \quad (18)$$

Here $\chi^{(\pm)}$ are the normalized internal eigenfunctions corresponding to the values $C=\pm 1$, respectively. They represent the state vectors $2^{-1/2}[|+-\rangle\pm|-+\rangle]$. All the other notations are identical to those used in discussing the 2π system. If kaons are bosons, the symmetry types associated with the lower subscripts are forbidden, and we find the well-known selection rule

$$(\text{Bose } K\bar{K}) \quad C=(-)^J. \quad (19)$$

This rule is characteristic of Bose kaons and could be used for testing the SP for kaons.

Since we will discuss $K^0\bar{K}^0$ decays in Sec. II in terms of the CP quantum number and the K_1 and K_2 decays, we include an explicit discussion of $K^0\bar{K}^0$ decays from this point of view. The only change we make in the discussion is to use the states $|K_1\rangle$ and $|K_2\rangle$, which are related by a unitary transformation to the states $|K^0\rangle$ and $|\bar{K}^0\rangle$.

We use the convention

$$|K^0\rangle=2^{-1/2}(|K_1\rangle+|K_2\rangle) \quad |\bar{K}^0\rangle=2^{-1/2}(|K_1\rangle-|K_2\rangle).$$

Then $\chi^{(\pm)}$ are represented by the following state vectors:

$$\chi^{(+)}=2^{-1/2}(|11\rangle-|22\rangle),$$
$$\chi^{(-)}=2^{-1/2}(|21\rangle-|12\rangle),$$

i.e., $\chi^{(+)}$ leads to K_1K_1 and K_2K_2 decays with the same probability, and $\chi^{(-)}$ leads to K_1K_2 decays. Therefore, the observation of the K decays, as is well-known, constitutes a straight measurement of the quantum number C, in spite of the fact that K_1 and K_2 are eigenstates of CP rather than C. Note in passing that

$$CP\Psi_{s,a}=\pm\Psi_{s,a}$$

as could be expected, since C and P are equivalent to the permutation of the internal and orbital quantum number respectively in this case. For Bose kaons, we have $CP=+$.

SP Selection Rules

Quite generally, in systems of two identical particles, we have definite relations between internal and orbital quantum numbers, which follow from SP and can be used for testing it. Relations (17) and (19) are particular cases of such relations. We list here the whole set of relations *which follow from SP with the usual connection between spin and statistics.*

Call X a particle and $\bar{X}$ its charge conjugate; let L denote the relative orbital angular momentum of the two-particle system, S its total spin (if any), and C its charge conjugation quantum number. We have

(i) for XX systems (e.g., $\pi^+\pi^+$, $\Lambda\Lambda$, pp):

$$(-)^L=(-)^S; \qquad (20)$$

(ii) for $X\bar{X}$ systems (e.g., e^+e^-, $\pi^+\pi^-$, $K_0\bar{K}_0$):

$$C=(-)^{L+S}. \qquad (21)$$

In the case of self-conjugate particles like γ or π^0, relation (21) still holds with $C=+1$. (Recall that $C_\gamma=-1$, $C_{\pi^0}=+1$.)

Call χ a strongly interacting particle with isospin t, $\bar{\chi}$ its G-conjugate.[12]
With the same notations as above and I for the total isospin, we have

(i) for $\chi\chi$ systems (e.g., $\pi\pi$, NN, KK):

$$(-)^L=(-)^{S+I+2t}; \qquad (22)$$

(ii) for $\chi\bar{\chi}$ systems (e.g., $\Sigma\bar{\Sigma}$, $K\bar{K}$, $N\bar{N}$):

$$G=(-)^{L+S+I}. \qquad (23)$$

Relation (23) still holds for self-G-conjugate particles like pions, in which case $G=+1$. (Recall that $G_\pi=-1$.)

Relations (20)–(23) are nonrelativistic versions of relations, which clearly hold even when the nonrelativistic approximation is not valid. We have exhibited them rather than the exact ones, because they make the subsequent arguments look more transparent. However, the reader should convince himself that relations expressing the fact that a two-particle system has a definite symmetry character hold in general and not only in the nonrelativistic limit. The exact relations are conveniently written in the helicity formalism.[13] They are obtained from the nonrelativistic ones by replacing

(a) $(-)^S$ by $(-)^{2s}H$,
(b) $(-)^L$ by P for XX and $\chi\chi$ systems, and by $(-)^{2s}P$ for $X\bar{X}$ and $\chi\bar{\chi}$ systems.

Here, s is the spin of each particle, P the parity of the two-particle state, and H its "helicity permutation operator," i.e., the operator which permutes the two helicity quantum numbers *and changes their sign* $(\lambda',\lambda''\to-\lambda'',-\lambda')$. The change in sign is due to the particular convention of sign taken in the definition of the helicity.

For example, relation (21) is the nonrelativistic version of the following exact relation for $X\bar{X}$ systems obeying SP:

$$PC=H. \qquad (24)$$

With more than two identical particles, we do not find as simple relations as those above any longer. The only property worthy of note is that the $I=0$ charge wave function of three isospin 1 particles is antisymmetric. Consequently, the orbital wave function of a $I=0$, 3π system must be antisymmetric if pions are bosons.

[12] L. Michel, Nuovo Cimento 10, 319 (1953). T. D. Lee and C. N. Yang, *ibid.* 13, 749 (1956). Recall that for the zero charge term of a $\chi\bar{\chi}$ multiplet, we have $G=C(-)^I$.
[13] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

II. EXPERIMENTAL EVIDENCE FOR SP AND POSSIBLE TESTS

From now on, SP will denote the symmetrization postulate taken with the usual connection between spin and statistics.

In this section, we review the experimental evidence in support of SP for the known species of particles, and investigate the possibility of performing experimental tests of it when they are needed.

There is no doubt that *electrons and nucleons are fermions*, and that *photons are bosons*. The evidence is particularly overwhelming in the case of electrons in view of the central role played by the Pauli principle in the dynamics of many-electron systems, i.e., atoms, molecules, and solids. For nucleons, the best evidence of all is given by the forbidden lines in the rotational spectra of homonuclear diatomic molecules, since they do not depend on the details of nuclear forces. The Bose nature of photons is revealed by the study of blackbody radiation and by the fact that quantum electrodynamics, in which the photon field is treated as a Bose field, quantitatively explains a wide range of electromagnetic phenomena.[14]

The situation with all the other known species of particles (π, strange particles, μ, neutrinos, etc. $\cdots$) is completely different. Large assemblies of identical particles cannot be produced in these cases. Furthermore, the dynamics of their interactions is still very poorly understood. In order to see whether SP holds or not, one has to resort to direct tests of the SP selection rules on systems containing two or, at most, three such identical particles. Up to now, for all these particles but pions, no such tests have been produced. In the case of pions, we benefit already from a rather large body of information about multipion systems. Most of it turns out to be inconclusive, but not all. From the results about the pionic decay modes of kaons, we have definite evidence that the multipion production obeys the SP selection rule. All this will come out of our discussion. We will also show that many significant tests of the SP selection rule can be devised. None of the tests that we propose look easy to realize in practice, but at least some of them in the cases of K and Λ, are well within the scope of present experimental technique.

This section is divided in three parts. The first part gives the principles and the general features of the tests. In the second part, we study, with some illustrative examples, the various types of tests that we may think of. The third part is devoted to a brief survey of the present experimental situation and of the possible tests for each species of particle.

[14] The Bose nature of photons is also expected from the fact that the electromagnetic field is measurable, and thus must commute with itself at space-like separation.

1. General Features of the Tests

The tests that we are looking for must be performed on systems containing very few identical particles of the species considered. In practice, except for pions, it is very hard to produce systems which contain more than two identical particles. In this section, we shall focus on two-particle systems. Much of what will be stated applies with obvious changes to systems containing more than two identical particles. The only observations which can be made on such systems are *one-body observations*, and are conveniently expressed, as shown by Eq. (5), as a sum of two terms, each of which correspond to a given symmetry type:

$$w(\lambda_1,\lambda_2)=|\langle\Psi|s\rangle|^2+|\langle\Psi|a\rangle|^2. \qquad (5)$$

The SP selection rule states that one of these terms must vanish.

An important feature of such observations—which was already noted about Eq. (5)—*is the absence of interferences between symmetry types*, a consequence of the indistinguishability of identical particles alone. This point should be kept in mind, since many experiments, which at first sight seem to provide acceptable tests, must be rejected on this ground.[14a] To illustrate this, consider a $\pi^+\pi^+$ scattering experiment. SP implies that the $\pi^+\pi^+$ system does not contain any odd waves. However, the forward-backward symmetry of the angular distribution, which is the usual criterion that waves of a given parity do not show up in a collision, is no test of SP, since it is only evidence that interferences between odd and even waves are not observed. In order to test SP, one could also think of exploiting the short range of the $\pi\pi$ force and the subsequent barrier effects at low energy. Let us assume that the energy is so low that only S, P, and D waves are expected to come in, the D-wave amplitude being an order of magnitude smaller than the P. Since the D wave interferes with the S wave, whereas the P wave does not, the P-wave contribution to the cross section has the same $\cos^2\theta$ dependence as the interference and the same expected order of magnitude. Again, we do not find any distinctive feature which permits to test the absence of P waves. In fact, without a detailed knowledge of the dynamics of the $\pi\pi$ interaction, there is absolutely no way of testing the Bose nature of pions in a $\pi^+\pi^+$ scattering experiment.

Another very crucial feature—which is also quite obvious from Eq. (5)—is that a set of *one-body measurements by itself does not give any information about symmetry types.* The determination of symmetry types unavoidably requires additional information about the observed two-particle system, which in turn requires a sufficient understanding of the dynamics of the production process. Thus, all tests of SP selection rules will necessarily imply some assumption about the interactions among particles. In order to be acceptable, a test must rely only on firmly established properties of the interactions.

Such well-established properties as the hierarchy of strength of the interactions (strong, electromagnetic and weak) and their respective invariance laws can always be safely assumed.[15] For strong interaction production processes—which represent the majority of the cases considered below—we may also safely assume that the forces have a short range. This leads to barrier effects, which set a limit on the complexity of the angular dependence of the wave at each energy, and suppress all partial waves but the S wave in the low-energy limit. Apart from their well-known invariance properties, the shortness of the range will be the only feature of the strong interactions which we assume in the tests discussed below. In the case of electromagnetic production, the assumption about the range has to be relaxed, but the weak coupling picture is valid and its consequences should be carefully explored. We shall also meet few but important cases involving weak interactions. Then, such assumptions as the PC invariance or the $|\Delta\mathbf{I}|=\frac{1}{2}$ rule are needed; the first of these assumptions is quite safe; the second is more questionable.

Going back to Eq. (5), we can see clearly now the way of building significant tests of the SP selection rule. Assume, for concreteness, that the two-particle system under observation is produced by strong interactions. As given by (5), the yield of the experiment is a sum of two terms, one SP obeying and one SP violating. We must look for situations where:

(i) the SP-obeying term is either forbidden by conservation laws, depressed by barrier effects, or exactly calculable,
(ii) the SP-violating term is *neither* forbidden by conservation laws *nor* suppressed by barrier effects.

Then, a nonvanishing yield for the SP violating term proves that SP is violated, and a zero yield is evidence that SP holds for this species. The two conditions above are quite crucial for the test to be significant. They will be referred to below as condition (i) and condition (ii).

2. Study of Various Types of Tests

To put some order in the discussion of possible tests, we tentatively classify them according to the way in which the set of identical particles can be produced. It

[14a] *Footnote added in proof.* In particular, the "tests of normal statistics for K mesons" by S. Barshay [Phys. Rev. **135**, B152 (1964)], which are based on the observation of interferences between dikaon states of opposite parity are in fact tests that the two kaons can be treated as identical particles with a charge degree of freedom (and that isospin is conserved). We thank Professor Barshay for an interesting correspondence on this subject.

[15] In particular, we assume isospin conservation in strong interaction production processes. Some care must be taken in the tests considered below, in which this assumption plays a role, since the deviations from the isospin conservation law due to electromagnetic effects look exactly like a small violation of the symmetrization postulate.

can be produced in a decay or in a collision. We further distinguish between simple production processes, in which the set is produced alone (e.g., $A \to XX$, $AB \to XX$), and associate productions, in which it is accompanied by one or several particles of a different species (e.g., $A \to CXX$, $AB \to CXX$). We consider first simple decays, then simple collisions, and finally associate production processes.

SP Forbidden Simple Decays: Evidences of the Bose Nature of Pions

Multipion systems can be produced in the decay of many unstable particles, i.e., $K^\pm$ and K^0, excited mesons (ρ, η, ω, etc.), and protonium. Dikaon systems can also be produced in this way. When the quantum numbers of the decaying particle are known, these may lead to significant tests of SP, if both conditions (i) and (ii) are met. This is best shown by treating specific examples. We shall treat here the decay of the ω, then the decays of kaons into pions.

The ω decay is a strong interaction process. The selection rule to be tested is relation (17).

The ω is an $I=0$ object. Its other quantum numbers are $J^{PG}=1^{--}$. Its 2π decay is forbidden by the SP selection rule. However, this cannot be taken as evidence for the rule, since it is also forbidden by G conservation and condition (ii) is not met. As for its 3π decay, there are two features which, at first sight, seem to be relevant to the SP selection rule for pions: (1) the sixfold sector symmetry of the density of events on a Dalitz plot whose three energy axes make 120° angles with each other, and (2) the regions of depletion of this Dalitz plot where one of the pions has its maximum possible energy. The first feature has nothing to do with Bose symmetry; it follows from the absence of interference between symmetry types, together with the $I=0$ property of ω. The second feature is also not a test of SP, because, contrary to conditions (i) and (ii) the SP-violating contributions are more depressed by barrier effects than the SP-obeying ones. Thus we conclude that, contrary to first appearance, no significant test of SP can be expected from consideration of the 3π decay of the ω either.[16]

[16] A more detailed analysis of the ω decay confirms this conclusion. Since the ω is 1^-, the Dalitz configuration (Ll) must have $L+l$ even, $L=l\pm1$ or $L=l$, and $L+l\neq0$. Thus, the only configurations allowed by **J**, P selection rules are those for which $L=l\neq0$ i.e., (11), (22), (33), (44), and so on. If pions are bosons, the 3π orbital wave function must be completely antisymmetrical, and the "even" configurations, (22), (44), etc., are forbidden. We see that the (00), whose absence is particularly easy to observe, is forbidden anyway by **J**, P conservation; the dominant contribution comes from the (11) term, whether pions are bosons or not. Furthermore, since, as a consequence of the indistinguishability of pions (and of **I** conservation), there is no observable interference between "odd" and "even" configurations, deviations from a pure (11) Dalitz plot come, in first approximation from the square of the (22) amplitudes and from the interferences between (11) and (33). It turns out that these two corrections have the same functional dependence in the Dalitz variables and cannot be distinguished. Thus, no practical test of the selection rule emerges from the study of the ω.

On the contrary, the decays of kaons into pions provide good evidences of the Bose nature of pions.[17]

Note first that for 2π systems, relation (21) leads to $PC=+$. Since we believe that weak interactions are PC-invariant and that the K_1^0 and K_2^0 are, respectively, $+$ and $-$, it follows that the decay of the K_2^0 into 2π is SP forbidden. Clearly, the decay is not forbidden by the conservation laws governing weak interactions. Condition (ii) is met, and the test is significant. Experiment shows that the K_2^0 does not decay into 2π. This can be taken as a very good evidence that π is a boson, together with a confirmation that weak interactions are PC-invariant.

For 3π decays of K^0, the situation is reversed. The lowest Dalitz configuration for a spin 0 ($\pi^+\pi^-\pi^0$) system is (00) for odd PC and (11) for even PC, in the case of Bose pions. This decay mode is therefore expected to have a smaller partial width for K_1^0, than for K_2^0, because of barrier effects (the ratio of partial widths can be estimated to be at least 10^{-2}). This, together with the typical (11) Dalitz configuration for the K_1^0 decay, would provide a significant test of the Bose nature of pions.

Another evidence of the Bose nature of pions is found in the decays

$$K^\pm \to \pi^\pm\pi^0, \tag{25}$$

$$K_1^0 \nearrow \pi^+\pi^- \tag{26a}$$

$$K_1^0 \searrow \pi^0\pi^0. \tag{26b}$$

Here the selection rule to be tested is again relation (17). It is found that the branching ratio of the charged to neutral decays of the K_1 is in accord with a pure $I=0$ final state[18] ($\simeq 2:1$) and that the $K^\pm$ decay rate is very much slower than that of K_1[19]: $r\equiv w(K^+\to\pi^+\pi^0)/w(K_1\to 2\pi)=1.4\times10^{-3}$. This is currently explained by the assumption that these weak decay processes are dominated by $|\Delta\mathbf{I}|=\frac{1}{2}$ terms. Then, the transitions to $I=2$ states are strongly depressed. From **J** conservation, we know that the dipions are produced in S states. Since the SP selection rule forbids $I=1$ for S states, we are left with a dominant $I=0$ in agreement with the experimental findings. Here again, the two conditions for significant tests of SP are met. The SP-obeying transitions are either forbidden (K^+ decay) or exactly calculable (K_1 decay), whereas the violating transitions are not forbidden by any further assumption. The weakness of the argument, however, is that the $|\Delta\mathbf{I}|=\frac{1}{2}$

[17] They have been recently discussed by H. C. von Baeyer, Phys. Rev. **135**, B189 (1964).

[18] This proves that pions are not fermions, regardless of whether or not the $|\Delta\mathbf{I}|=\frac{1}{2}$ rule holds. See B. S. Thomas and W. G. Holladay, Phys. Rev. **110**, 981 (1958) for a similar argument based on the τ decay. See also the discussion of reaction (29) below.

[19] M. Roos, Nucl. Phys. **52**, 1 (1964).

rule for weak interactions, which is anyway an approximate rule, is not yet very firmly established.

Simple Two-Body Collisions: Tests That Λ Are Fermions

Next in simplicity for producing systems of identical particles, we find the simple two-body collisions, e.g.,

$$\bar{p}p \rightarrow \bar{\Lambda}\Lambda\,, \tag{27}$$

$$\Xi^- p \rightarrow \Lambda\Lambda\,. \tag{28}$$

In practice, apart from reaction (28), all of them are reactions in which a charge-conjugate (or G-conjugate) pair goes into another charge-conjugate (or G-conjugate) pair[20]:

$$\bar{A}A \rightarrow \bar{X}X\,. \tag{29}$$

However, the possibility of building tests on reactions of type (29) is very much restricted as a consequence of the following lemma.

Lemma: Consider reaction (*29*) *where an SP obeying charge-conjugate pair* $\bar{A}A$ *produces another charge-conjugate pair, which may not obey* SP. *Then, SP violating transitions are absolutely forbidden by* **J**, *P*, *and C conservation* (a) *if both A and X have spin* 0, (b) *if one of them is a Dirac spin-*$\frac{1}{2}$ *particle and the other has spin* 0. *If both A and X are Dirac spin-*$\frac{1}{2}$ *particles, then SP-violating transitions are those in which the total spin*[21] *is not conserved* ($\Delta S\neq 0$); *they are forbidden if one of the pairs is in an S state.*

The same results hold as a consequence of **J**, *P*, *C*, *and* **I** *conservation for G-conjugate pair to G-conjugate pair reactions.*

The proof is straightforward. Here, we consider only the case of two charge-conjugate spin-$\frac{1}{2}$ pairs and leave the other cases to the reader. Since they are assumed to be Dirac particles, the parity of each pair is opposite to that of its relative orbital angular momentum $[P=(-)^{L+1}]$. Thus, conservation of P and C implies conservation of $C(-)^L$. Since the $\bar{A}A$ pair obeys the SP selection rule (21), whereby $C(-)^L=(-)^S$, violation of SP in $\bar{X}X$ is equivalent to a change of sign of $(-)^S$ in the transition, i.e., $\Delta S\neq 0$. Because of **J** and P conservation, these singlet to triplet and triplet to singlet transitions are allowed only if $J=L$ in the initial and the final state, and they are forbidden when one of the L vanishes (S state), Q.E.D.

An obvious consequence of the lemma is that condition (ii) has no chance to be met[22] unless both A and X have spin $\frac{1}{2}$. Thus, no valid test of the Bose nature of π can be based on the analysis of reactions of this type; on the other hand, the fact that reactions of this type are observed, e.g., $\bar{p}p\rightarrow\pi^+\pi^-$, insures that pions are not fermions. In the same way, from the occurrence in nature of the reaction $\bar{p}p\rightarrow K^0\bar{K}^0$, we conclude that kaons are certainly not fermions, but no reaction of this type may lead to any valid test of the Bose nature of K. Better, if we assume that π are bosons, we cannot find any test of the Bose nature of K in πN collisions (e.g., $\pi^-p\rightarrow\Lambda K^0\bar{K}^0$) if the collisions are known to be peripheral, since, in this case, the effective $K\bar{K}$ producing process is $\pi\pi\rightarrow K\bar{K}$.

We focus now on spin-$\frac{1}{2}$ particles. The tests, if any, should involve a measurement of the total spin S of the two-particle system. Therefore, in practice, they apply only to particles like Λ, μ, whose polarization can be obtained from the angular correlation of their decay products. We shall discuss first reaction (27), as an illustrative example of reactions of type (29), then reaction (28).

With reaction (27) our aim is to look for spin-nonconserving transitions. The test will consist in preparing the $\bar{p}p$ pair in a pure triplet state and looking whether the production of singlet $\bar{\Lambda}\Lambda$ pairs is forbidden or not.

The pure triplet initial state is obtained by taking the $\bar{p}$ beam and the p target completely polarized along some direction Oy perpendicular to the incident momentum. In practice, it is possible to have only partially polarized beam and target. With partial rather than complete polarization, the situation is somewhat more complicated, but the procedure remains essentially the same. We shall not go here into these complications and shall assume complete polarization.

Now, the cross sections σ_s and σ_t for the production in a given direction of singlet and triplet $\bar{\Lambda}\Lambda$ pairs respectively, are given by[23]

$$(\bar{\Lambda}\Lambda \text{ pairs}) \qquad \begin{aligned} \sigma_s&=\tfrac{1}{4}\sigma\left[1+\frac{9}{\alpha^2}\langle\cos u\rangle\right] \\ \sigma_t&=\tfrac{3}{4}\sigma\left[1-\frac{3}{\alpha^2}\langle\cos u\rangle\right]. \end{aligned} \tag{30}$$

These formulas hold whether the initial $\bar{p}p$ state is polarized or not. σ is the corresponding total production cross-section along this direction, $\alpha(\simeq -0.66)$ is the asymmetric decay parameter of the Λ, and u is the angle between the two decay-pion momenta taken, respec-

---

[20] Here, we assume $A\neq X$. There are simple tests based on scattering experiments, i.e., $XX\rightarrow XX$ or $\bar{X}X\rightarrow\bar{X}X$. Thus, in the charge-exchange collision $K^+K^-\rightarrow\bar{K}^0K^0$, SP forbids the production of K_1K_2 pairs at 90° c.m. angle. We do not go into this, since these experiments are unfeasible with present techniques.

[21] For clarity we use all along the nonrelativistic language and speak of total spin, rather than the correct language with the H operator.

[22] We assume here, as is always the case in practice, that one of the particles, A say, is known to obey SP. If the nature of both A and X is questioned, we may find significant tests that at least one of them obey the postulate. For example, the decay of the S-state protonium into $2K_1$ (or $2\pi_0$) is forbidden if either p or K (or π) obey the postulate. If this decay were observed, we would have to conclude that *both* p and K (or π) violate SP providing that we are sure that the protonium is in an S state. Since it is not observed, we are sure that at least one of the two species of particles obey SP and that the protonium is in an S state.

[23] C. Cohen-Tannoudji and A. M. L. Messiah, Nuovo Cimento 23, 853 (1964). See in particular Note 7.

TABLE I. Selection rule for the reaction $\Xi^- p \to \Lambda\Lambda$ at low energy.

η	SP forbidden	SP allowed	Resulting angular correlation
+	${}^3S_1 \to {}^3S_1, {}^3D_1$	${}^1S_0 \to {}^1S_0$	$\langle \cos u \rangle = -\frac{1}{9}\alpha^2$
−	${}^3S_1 \to {}^1D_1$	${}^3S_1 \to {}^3P_1$ ${}^1S_0 \to {}^3P_0$	$\langle \cos u \rangle = \frac{1}{9}\alpha^2$

tively, in the rest frames of their parent particles. The symbol $\langle \cdots \rangle$ denotes the average over the directions of the two decay pions. Thus, the singlet and triplet contributions can be separately measured by means of a suitable angular correlation experiment.

With the pure triplet initial state taken above, σ_s must vanish if the Λ is a fermion, i.e., the correlation coefficient $\langle \cos u \rangle$ must be equal to its lowest bound $-\alpha^2/9$.

Let us now discuss condition (ii). The only conservation law which might come into play is the one associated with the reflection R around the production plane, when the latter is normal to Oy. Then, the initial state is an eigenstate of R ($R=+$) and the conservation of R is equivalent to the conservation of the parity of m_S, component of the total spin along Oy. This requires $m_S=+1$ or -1 for the $\bar{\Lambda}\Lambda$ pair, which excludes singlet production. Besides this, barrier effects will suppress all but S waves in the vicinity of the threshold, which also excludes triplet to singlet transitions according to the lemma. Thus, condition (ii) is met if the energy is sufficiently far above threshold, so that higher waves are produced in the final state, and if the normal to the production plane is sufficiently far away from the initial direction of polarization.

We turn now to reaction (28). It has already been studied in the literature for information about the (Ξp) parity.[24] We consider here the absorption in flight. The K-shell absorption of a $\Xi^- p$ atom leads to similar results.

Like for $\bar{\Lambda}\Lambda$ pair production, the singlet and triplet contributions can be separated by angular correlation. The expressions for σ_s and σ_t are those given by Eqs. (30) with $-\alpha^2$ instead of α^2 everywhere. The reaction is exoergic ($Q\simeq 29$ MeV) and conserve parity. Therefore, *in the low-energy limit*, all waves but the S wave are suppressed by barrier effects in the initial state and the parity of the $\Lambda\Lambda$ pair is equal to the intrinsic parity η of the (Ξp) system. If the Λ is a fermion, $(-)^S=\eta$, that is $\sigma_t=0$ for $\eta=+$ and $\sigma_s=0$ for $\eta=-$. The operation of the SP selection rule is summarized in Table I. Clearly the forbidden transitions are forbidden by SP, and not by any conservation law or barrier effects. Thus, condition (ii) is met and the test is significant.

[24] L. B. Okun', I. Ia. Pomeranchuk, and I. M. Shmushkevich, Zh. Eksperim. i Teor. Fiz. 34, 1246 (1958) [English transl.: Soviet Phys.—JETP 7, 862 (1958)]; S. B. Treiman, Phys. Rev. 113, 355 (1959).

Associate Production: Application to Kaons

We now discuss possible tests in the more complicated cases when the system of two identical particles is produced in a reaction together with one or several other particles, that is reactions of the type $A \to CXX$ or $AB \to CXX$. Here, X denotes a particle of the species under study, without any restriction on the values of its internal variables. Thus XX stands for what was denoted by XX, $X\bar{X}$, $\chi\chi$ or $\chi\bar{\chi}$ in the Sec. I.4. The tests, which can be thought of in these more complicated cases are very numerous and diverse. But they look much more difficult than those using simple production processes, because they always demand detailed information about a certain selection of events, the yield of which is expected to be quite small. The tests, which we give below and in the Appendix, for illustration, may not be the easiest to perform in this wide collection. But their description should give a clear enough picture of the different types of test entering this category, and of the way one can be convinced of their respective significance.

The SP selection rule to be tested will be relation (20), (21), (22), or (23), according to the case. One common feature of these relations is that they relate the orbital parity $(-)^L$ to internal variable quantum numbers. The tests will consist in producing the XX system with well chosen values of the internal quantum numbers, and then to find a way of checking that the waves having the SP forbidden parity are actually missing. For this purpose, we have to make use of some observable effect characterizing states with no odd waves, or states with no even waves. To be of any value to us, this effect must be quite general; i.e., its occurrence should not depend on more detailed features of the dynamics of the production process than those that we are willing to admit and that were indicated in Sec. II.1.

To our knowledge, three effects may be of use and lead to three different types of test. The first two have to do with the vanishing of orbital waves of given parity at suitable angles. First, orbital waves with a given value M of the magnetic quantum number vanish at right angle to the quantization axis if their parity is opposite to $(-)^M$. Second, orbital waves with a given value, + or −, of the quantum number R associated with the reflection through a given plane, vanish at right angle to this plane if their parity is opposite to R. The last useful effect (which works only for strong interaction processes) is the suppression of all but the S-wave contribution when the invariant mass m_{XX} of the XX system approaches its threshold value.

Let us discuss first the tests of SP using waves of given M ("M test").

We recall that the spherical harmonic $Y_L{}^M(\theta,\varphi)$ vanishes (like $\cos\theta$) when θ goes to $\frac{1}{2}\pi$ if $(-)^{L+M}$ is odd, whereas it keeps a finite value if $(-)^{L+M}$ is even. In particular, we find the following results[25] about the emission at $\theta=\frac{1}{2}\pi$:

(a) In the absence of even-L waves, the contribution of $M=0$ characteristically vanishes like $\cos^2\theta$.

(b) In the absence of odd-L waves, the contribution of $M=\pm1$ characteristically vanishes like $\cos^2\theta$.

(c) No characteristic effect shows up in the interference terms between $M=0$ and $M=\pm1$ when either even- or odd-L waves are missing.

Consequently, in states for which *even-L waves are SP forbidden*, a possible test will consist in separating the $M=0$ component along some quantization axis and checking that its contribution in the emission at right angle of this axis vanishes like $\cos^2\theta$. In states for which *odd L waves are SP forbidden*, the same treatment has to be applied to the $M=\pm1$ component rather than $M=0$. In proposing such tests, one must always make sure that condition (ii) is actually met. We illustrate this on the reaction

$$K^-p\rightarrow\Lambda K\bar{K}. \tag{31}$$

As noted already, even- and odd-C contributions in the $K\bar{K}$ state can be separated by observing the decay mode of the kaons. According to the SP selection rule (19), K_1K_1 (and K_2K_2) pairs must not occur in odd orbital states, K_1K_2 systems must not occur in even orbital states. To test this, we must manage to separate $M=0$ and $|M|=1$ dikaon waves. This can be done by selecting events in which the dikaon is emitted along the line of flight of the incident kaon (z direction). The desired $|M|=1$ and $M=0$ states correspond respectively to events in which the z component of the baryon spin does flip and does not flip. There is unfortunately no way of separating these states with an unpolarized target.[26]

Let us assume that the protons are fully polarized along the z axis ($M_p=+\frac{1}{2}$). Call, respectively, $(\theta_K\varphi_K)$ and $(\theta_\pi\varphi_\pi)$ the spherical angular coordinates of the direction of K^0 emission in the rest system of the dikaon and of the direction of the decay π in the rest system of the Λ. With obvious notation, the forward production amplitude of the dikaon, can be written:

$$\begin{aligned}A_+&=\text{no spin flip}+\text{spin flip}\\&=\Lambda_+(K\bar{K})_0+\Lambda_-(K\bar{K})_1.\end{aligned}$$

Squaring, then averaging over φ angles and over the polarization of the decay nucleon of Λ, we obtain the following form for the forward cross section:

$$\sigma_+=(1+\alpha\cos\theta_\Lambda)|F_0(\theta_K)|^2+(1-\alpha\cos\theta_\Lambda)|F_1(\theta_K)|^2.$$

[25] For $L+M$ even and $|M|=0$ or 1, $|Y_L{}^M(\frac{1}{2}\pi,\varphi)|^2$ is found to be 1.25 times its average value $\frac{1}{4}\pi$ with a 5% accuracy. Thus unless cancellations occur through interference of different values of L, the contribution from $(L+M)$ even is *a priori* quite large at $\theta=\pi/2$.

[26] With an unpolarized target, one can separate the contribution of the interference term between $M=0$ and $M=1$ states by measuring the azimuthal correlation of the K^+ and of the decay pion of the Λ. As noted above, however, this is of no use for testing SP.

α is the asymmetry parameter of the Λ decay, F_0 and F_1 are, to within inessential factors, the transition amplitudes corresponding to $M=0$ and $M=1$ waves, respectively. More generally, the forward cross section σ_P corresponding to target protons having polarization P along the z axis, reads:

$$\sigma_P=\sigma_0(1+\alpha PG\cos\theta_\Lambda)$$

with

$$G(\theta_K)=\frac{|F_0|^2-|F_1|^2}{|F_0|^2+|F_1|^2}.$$

Thus, a measure of the ratio of the polarized to the unpolarized cross sections, permits the determination of the angular correlation function $G(\theta_K)$, which in turn lead to the relative contribution of $M=0$ and $M=1$ waves through the formulas

$$\frac{|F_0|^2}{|F_0|^2+|F_1|^2}=\tfrac{1}{2}(1+G),\quad \frac{|F_1|^2}{|F_0|^2+|F_1|^2}=\tfrac{1}{2}(1-G).$$

If kaons are bosons, odd waves are forbidden for K_1K_1 pairs. Then, for ΛK_1K_1 events, $|F_1|^2$ will vanish like $\cos^2\theta_K$ at $\theta_K=\frac{1}{2}\pi$, whereas $|F_0|^2$ will remain finite. For ΛK_1K_2 events, the converse result will apply, $|F_0|^2$ going to zero whereas $|F_1|^2$ remains finite. Hence the test we are looking for. As is readily seen by inspection, **J** and P conservation do not forbid the waves which are forbidden by the SP rule, but $L=1$ and higher waves are depressed by barrier effects for events in which the invariant mass m_{KK} is not far enough from its minimum value $2m_K$. Thus, condition (ii) is always expected to hold in the analysis of ΛK_1K_2 events. In the analysis of ΛK_1K_1 events, however, one should keep away from the low invariant mass limit, and check that higher waves than $L=0$ are actually present, by observing the deviation from spherical symmetry in the angular distribution of the kaons in their center-of-mass system.

We draw the reader's attention to the similarity of the procedure followed in these tests with the analysis of Adair or its generalized versions.[27] The choice of the range of solid angle toward the forward direction in which the dikaon is collected for this type of experiment is made exactly like in the Adair analysis ($\theta\lesssim\lambda/R$) and for the same reason.

We turn now to the second type of tests ("R tests").

It is performed on waves with a given value of the "plane reflection" quantum number R, and uses the known property of spherical harmonics that waves, which have a parity opposite to R, characteristically vanish (like $\sin\theta$) at right angle to the plane of symmetry.

A very simple illustration of the possible use of this

[27] R. K. Adair, Phys. Rev. **100**, 1540 (1955); S. B. Treiman, *ibid.* **128**, 1342 (1962); C. Itzykson and M. Jacob, Phys. Letters **3**, 153 (1963).

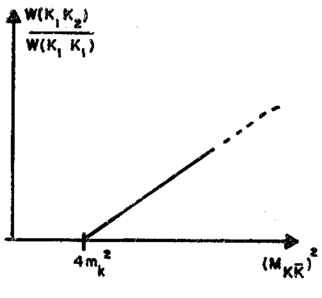

FIG. 1. Ratio of cross sections for K_1K_2 and K_1K_1 events versus mass squared of the $K\bar{K}$ system near threshold for Bose kaons.

for testing SP is given by the reaction

$$K^+ \mathrm{He}^4 \to H_\Lambda{}^4 K^+ K^+. \qquad (32)$$

The plane of symmetry to be considered here is the plane of production of the dikaon, i.e., the plane parallel to the incident direction and to the direction of emergence of the center of mass of the K^+K^+ system. R is the quantum number associated with the reflection through this plane. It is known that K^+ is a pseudoscalar, and that the hyperfragment $H_\Lambda{}^4$ is a scalar particle like He^4. Since R is conserved in this strong interaction process, we find that the dikaon produced in the final state has $R=-$. If kaons are bosons, this value is opposite to the allowed parity of the dikaon wave and the emission of K^+ at right angle of the production plane is forbidden. To be significant, the test should be performed with dikaons emitted sufficiently far away from the forward (or backward) direction and with an invariant mass m_{KK} sufficiently far above its threshold value $2m_K$, in order that odd dikaon waves be not forbidden by conservation laws or barrier effects [condition (ii)].

As a second illustration, let us go back to reaction (31). Again R will be the quantum number associated with the reflection through the production plane of the $K\bar{K}$ system. To build a test of SP, polarization of the target is still needed, but now it has to be perpendicular to the production plane. For simplicity, we assume complete polarization of the proton target in this direction. From angular correlations of the decay products of the Λ, it is easy to separate the final states with spin flip (i.e., spin of the Λ opposite to that of the proton) from those without spin flip. Since R is conserved, dikaons produced with spin flip have $R=+$, whereas those produced without spin flip have $R=-$. For Bose kaons, K_1K_1 (and K_2K_2) pairs emitted at right angle to the production plane are necessarily associated with spin flip transitions, whereas K_1K_2 emitted along the same direction are necessarily associated with non-spin-flip transitions. The discussion of condition (ii) is similar and gives the same result as in the "M test" discussed above.

We turn finally to the third type of tests. It uses the barrier effect which, in strong interaction production processes, suppresses all but S-wave contributions in the limit $m_{XX}\to 2m_X$. For this purpose, we have to select quantum numbers of the XX system associated with odd waves by the SP selection rule, and make sure [condition (ii)] that the S-wave production of XX is not also forbidden by conservation laws. Then a significant test that the SP selection rule holds, consists in observing if the production cross section vanishes like $(m_{XX}-2m_X)$ times the appropriate phase factor when $m_{XX}\to 2m_X$.

To illustrate this, let us look again into reaction (31). We are led to compare the two channels

$$K^-p \nearrow \Lambda K_1K_1 \qquad (31a)$$

$$K^-p \searrow \Lambda K_1K_2. \qquad (31b)$$

The S-wave production is forbidden in the second channel, if kaons are bosons, whereas the first channel remains allowed. We do not see any conservation law to forbid S waves in any channel. Better, the cross section for reaction (31a) can be used to some extent as a good measure of the order of magnitude to be expected for reaction (31b), if the SP selection rule does not operate. Thus, a significant test that kaons are bosons is obtained by checking that:

(i) no relative S wave occurs in the K_1K_2 pair, i.e., calling $\mathbf{q}$ the relative momentum, the square of the matrix element vanishes at least as fast as q^2 when $q\to 0$, or equivalently as $m_{K_1K_2}-2m_K$ when $m_{K_1K_2}\to 2m_K$;

(ii) channel (31a) is much more copious than channel (31b) in this limit.

Figure 1 illustrates the expected threshold behavior, in the case of Bose kaons, of the ratio of the cross section for K_1K_2 events to that of K_1K_1 events, as a function of the square of the invariant mass of the $K^0\bar{K}^0$ system.

In practice, occurrence of $K\bar{K}$ resonances in both the K_1K_2 (i.e., ϕ_0 production) and K_1K_1 channels close to threshold may make this experiment more difficult than could be expected at first sight.

3. Survey of the Experimental Situation

In the light of the above discussion, we now make a survey of the experimental situation for species of particles other than electrons, nucleons and photons.

Pions

The nonoccurrence of SP forbidden decays for kaons, as mentioned above, can be taken as a conclusive evidence of the Bose nature of pions, especially the observed selection rule against the 2π decay of the $K_2{}^0$.

We have looked systematically for other evidence, and have not found anything but rather complicated tests. We first have reviewed the decay modes of all the known unstable particles.[19] As shown above, nothing significant emerges from the decay of the ω; we reach the same conclusion with the η and the ρ. For all other excited mesons, the situation is inconclusive because of uncertainties in the spin assignment.[28] The case of the protonium is discussed in the Appendix. We find a possibility of test there, but it looks quite difficult. Possible tests using associate production are also discussed in the Appendix.

Kaons

For kaons, as well as for all species considered below, we do not find any evidence of the SP selection rule on the record.

In fact, there does not seem to be any easy test available for kaons. We have seen that simple two-body collisions cannot lead to any test. For the same reason, the simple decay of the protonium cannot be, and is found not to be, SP forbidden. For all other unstable particles, e.g., the ϕ which decays into $K_1^0K_2^0$, the situation is still inconclusive, pending a measurement of the spin. Unless an example of SP-forbidden simple decay is found, one will very probably have to resort to associate production processes, in order to test significantly the Bose nature of kaons.

The three types of tests in associate production can be applied to reaction (31), as shown above. The M and R tests can also be applied in a very similar fashion to the reaction

$$K^+p \rightarrow \Lambda K^+K^+. \tag{33}$$

The difficulty with M and R tests in reactions (31) and (33) is that they require suitably polarized proton targets. Tests of this type, which do not require this rather difficult technique, may be worth noting, e.g., the R test with reaction (32) described above, and the M test with the protonium decay described in the Appendix. Examples of tests using the low-mass limit in other reactions than (31) are also listed in the Appendix.

Hyperons

Two remarks are in order with hyperons. First, simple two-body collisions are no longer *a priori* excluded for tests, and they should provide the easiest ones. Second, a test will always imply a measurement about the spin state of the hyperon pair, and there is no practical way of doing it, unless the hyperon has an asymmetric decay. Thus, we restrict ourselves to hyperons having an asymmetric decay (Λ, Σ^+, etc., but not Σ^-), and look for tests of their Fermi nature in simple two-body collisions.

The case of the Λ hyperon is the most favorable. We find two reactions, which provide significant tests, i.e., reactions (27) and (28). This has been studied in detail above. The experiment with reaction (28) looks *a priori* easier, since it does not require a polarized proton target.

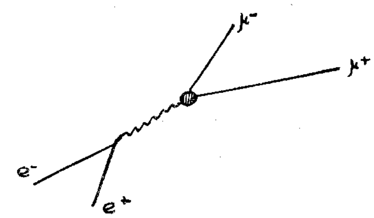

FIG. 2. Dominant diagram in $e^+e^- \rightarrow \mu^+\mu^-$.

We do not find in nature any reaction like (28) with other hyperons. But reactions like (27) do exist for all of them, and lead in the same way to practical tests of the Fermi nature for all the hyperons having an asymmetric decay.

Muons

Since muons also have an asymmetric decay, we meet at first sight the same favorable situation for tests as for the Λ. In particular, one might expect the study of the reaction

$$e^+e^- \rightarrow \mu^+\mu^- \tag{34}$$

to provide a very direct means of testing the Fermi nature of muons, exactly in the same way as reaction (27) provided a test for the Λ.

Here, however, the production process is due to electromagnetic rather than strong interactions. Therefore, the validity of condition (ii) has to be reinvestigated.

The main point to make is that the coupling is weak, and that the dominant contribution to the production amplitude comes from the one-photon exchange diagram (Fig. 2). The corresponding term leads to a $\mu^+\mu^-$ state, which has the same transformation properties under inhomogeneous Lorentz transformation and under charge conjugation as the virtual photon, i.e., $J^{PC}=1^{--}$. Remembering that the intrinsic parity of a pair of charge conjugate Dirac particles is odd, we conclude easily that this state is a triplet with even orbital angular momentum (or, more correctly, $HP=-$); therefore, the SP selection rule (21) is necessarily verified, whether the μ is a fermion or not. Condition (ii) is not fulfilled.

In conclusion, to the lowest order in $e^2/\hbar c$, reaction (34) does not provide any significant test of the Fermi nature of muons. Possible deviations from the SP selection rule come from higher order terms and can be displayed only by high precision measurements, which are hardly realizable in practice.

The same remark applies to all production processes through the exchange of one virtual photon, whether simple or associate. Thus, contrary to expectation, we do not find any practically feasible test of the Fermi nature of muons.

[28] To be of use for this purpose, the spin assignment should not be based on the symmetrization postulate. Thus the case of f, to whom the spin 2 has been attributed on the ground that odd spins are excluded by Bose statistics [L. Bondar *et al.*, Phys. Letters **5**, 153 (1963)], should be discarded. The same remarks apply to the ϕ [P. L. Connolly, E. L. Hart, K. W. Lai, G. London, G. C. Moneti *et al.*, Phys. Rev. Letters, **10**, 371 (1963)].

APPENDIX: DISCUSSION OF SOME SP TESTS FOR PIONS AND KAONS

η Decay

No test of the SP selection rule for pions emerges from the analysis of the decay of the η.

The η has $J^{PC}=0^{-+}$ and $I=0$. $\eta\to 2\pi$ is forbidden by $\mathbf{J}$ and P conservation, $\eta\to 3\pi$ cannot be a strong interaction process because of G conservation. Let us see whether the C conserving electromagnetic processes

$$\eta\to\pi^+\pi^-\pi^0, \tag{A1}$$

$$\eta\to\pi^+\pi^-\gamma, \tag{A2}$$

could be used for testing the SP selection rule $PC=+$ in the thereby produced $\pi^+\pi^-$ system. With the z axis taken along the line of flight of the π^0 or the γ, one selects $(\pi^+\pi^-)$ pairs with given values of M. Unfortunately, the M test cannot be applied, because the selected values of M, i.e., $M=0$ for (A1) and $|M|=1$ for (A2), are precisely not those which are useful for the M test. In the decay (A2), where SP forbids even waves, one could think of testing the absence of S wave in the low-mass limit. But here again, condition (ii) is not met, because the S-wave production is forbidden anyway by the $0\to 0$ selection rule in electromagnetic processes.

Protonium Decay

Since the simple decay processes $(\bar{p}p)\to 2\pi$ or $(\bar{p}p)\to\bar{K}K$ cannot provide any test as a consequence of our lemma, we have to look into the decay modes into three or more particles.

The decay of the protonium is known to occur from the $K-$ shell, i.e., from the two levels $J^{PC}=0^{-+}$ and 1^{--}. The difficulty in building tests there is that most often the respective contributions from these two levels cannot be measured separately.

We have not found any test for Bose pions from the analysis of the 3π decay modes, i.e., $(\bar{p}p)\to 3\pi^0$ and $(\bar{p}p)\to\pi^+\pi^-\pi^0$. More complicated decay modes provide tests, which look hardly feasible. For example, consider the 4π decay mode:

$$(\bar{p}p)\to\pi^+\pi^-\pi^+\pi^- \tag{A3}$$

and select the events, when the four emitted π are coplanar. Call R the operator of reflection through this plane. Since $R=+$ for these events, $J=0$ is excluded, hence $C=-$. For Bose pions, the two $(\pi^+\pi^-)$ pairs must have an opposite parity. They cannot be both in an S state at the same time. This absence of S-wave can be tested by observing the yield in the limit, when the invariant masses of both pairs simultaneously go to their threshold value.

Turning to kaons, we look for tests using decay modes of the type $(\bar{p}p)\to C\bar{K}K$. The interesting cases turn out to be

$$(\bar{p}p)\to\omega K_1K_2, \tag{A4}$$

$$(\bar{p}p)\to\rho^0K_1K_2. \tag{A5}$$

We discuss the mode (A4). The same arguments apply to (A5), since the ω and the ρ^0 have the same J^{PC} quantum numbers, i.e., 1^{--}. The conservation of C implies $J^{PC}=0^{-+}$ for the initial state. Then, take the direction of emission of the ω for the quantization axis Oz. It is possible, from angular correlations of the decay products of the ω, to separate the contribution $M_\omega=0$ from the contribution $|M_\omega|=1$. Selecting $M_\omega=0$, we obtain a $M=0$ state for the K_1K_2 system. Then, the SP selection rule (19) forbids the kaon emission at right angle to the quantization axis. Inspection shows that the conservation laws forbid the S wave, but not the D wave in this dikaon production, and that barrier effects, although favoring P-wave emission, should not suppress very much the D wave. If kaons were not bosons, the K_1K_2 system should thus exhibit a D-wave contribution, especially at right angle to the z axis, where it would be the dominant contribution. Thus, we are in a case where condition (ii) is reasonably well fulfilled.[29]

Associate Production of Pions ($AB\to C\pi\pi$)

We have not found any M or R test using a reaction which involves only spinless particles. Consider, for example the reaction

$$\pi^\pm\,\mathrm{He}^4\to\mathrm{He}^4\pi^\pm\pi^0. \tag{A6}$$

The dipion is produced in an $I=1$ state, and even dipion waves are SP forbidden. However, since the π is pseudoscalar, the dipion is $R=-$, hence even dipion states do not contribute to the emission of pions at right angle of the plane of production, as a consequence of $\mathbf{J}$ and P conservation alone; this excludes the R test. A similar argument excludes the M test. Significant M and R tests can be built only with reactions involving spin-$\frac{1}{2}$ particles, like $\pi^+p\to n\pi^+\pi^+$, and require polarization measurements, which make them unpractical.

The only practical tests in sight are those based on the low-mass limit. They consist in producing a dipion in a $I=1$ state, and testing the absence of S wave through observations near threshold. This can be done, for example, with the reactions

$$\pi^+p\nearrow n\pi^+\pi^+ \tag{A7a}$$

$$\pi^+p\searrow p\pi^+\pi^0. \tag{A7b}$$

The amplitudes T_S and T_a for the symmetric and anti-

[29] The decay into ωK_1K_1 is known to be copious. The same is expected for the decay into ωK_1K_2; it is unfortunately rather difficult to observe.

symmetric $N\pi\pi$ final states can be written

$$T_{S,a}=(1/\sqrt{5})\{2(n++)-(1/\sqrt{2})[(p+0)+(p0+)]\}\times A_{s,a}^{(2)}(\mathbf{k}',\mathbf{k}'')+(1/\sqrt{2})[(p+0)-(p0+)]\times A_{a,s}^{(1)}(\mathbf{k}',\mathbf{k}'').$$

The coefficients of the A's are the normalized charge wave functions, and correspond to the $I=2$ and $I=1$ two-pion contributions, respectively. The momenta $\mathbf{k}'$, $\mathbf{k}''$ are associated with the pion charges in the same order. The subscripts on the A's refer to the symmetry under exchange of the momenta. The A's also depend on additional variables which are not relevant here. The probability $w(\mathbf{k}'+,\ \mathbf{k}''+)$ for observing reaction (A7a) with momenta $\mathbf{k}'$, $\mathbf{k}''$ for the two π^+ is, up to irrelevant factors,

$$w(\mathbf{k}'+,\mathbf{k}''+)=\tfrac{4}{5}[|A_s^{(2)}(\mathbf{k}',\mathbf{k}'')|^2+|A_a^{(2)}(\mathbf{k}',\mathbf{k}'')|^2].$$

The corresponding probability for reaction (A7b) with momenta $\mathbf{k}'$ for π^+ and $\mathbf{k}''$ for π^0 is

$$w(\mathbf{k}'+,\ \mathbf{k}''0)=\tfrac{1}{10}[|A_s^{(2)}-\sqrt{5}A_a^{(1)}|^2+|A_a^{(2)}-\sqrt{5}A_s^{(1)}|^2],$$

where the A's should be taken at the values $\mathbf{k}'$, $\mathbf{k}''$ of their arguments. Here condition (ii) is obviously satisfied, so we isolate the pure $I=1$ contribution to reaction (A7b), according to the formula

$$w_1(\mathbf{k}'+,\ \mathbf{k}''0)=w(\mathbf{k}'+,\ \mathbf{k}''0)+w(\mathbf{k}''+,\ \mathbf{k}'0)-\tfrac{1}{4}w(\mathbf{k}'+,\ \mathbf{k}''+)=|A_a^{(1)}|^2+|A_S^{(1)}|^2.$$

If pions are bosons, w_1 must vanish like $q^2\equiv(\mathbf{k}'-\mathbf{k}'')^2$ times the appropriate phase space factor when $q\rightarrow 0$. In practice, this means that w_1 must become negligibly small compared to w_{+0}, w_{++} or w_2 in that limit.

One can imagine many other reactions like (A7) leading to tests based on the same property, namely the absence of S wave in $I=1$ dipion states. Of particular interest are reactions of this type, where the two pions are produced in a *pure* $I=1$ state, e.g.,

$$\pi^{\pm}d\rightarrow d\pi^{\pm}\pi^0\,, \tag{A8}$$

$$p\,\mathrm{He}^3\rightarrow \mathrm{He}^4\pi^+\pi^0. \tag{A9}$$

Note that reaction (A6) does not lead to any test because dipion S states are forbidden by **J** and P conservation alone.

Other cases of interest are those in which three pions are produced in a pure $I=0$ state, e.g.,

$$dd\rightarrow \mathrm{He}^4\pi^+\pi^-\pi^0. \tag{A10}$$

Relative S wave, in each pair of pions, which should dominate in their respective low-mass limit, are forbidden if pions are bosons.

Associate Production of Kaons

Tests of the absence of S wave in dikaon systems in the low-mass limit may apply either to $K\bar{K}$ in the K_1K_2 channel, or to KK in the $I=0$ channel. A typical example of the first case is given by reaction (31). Examples of the second case are given by the reactions

$$K^+p\nearrow\Sigma^0K^+K^+ \tag{A11a}$$

$$K^+p\searrow\Sigma^+K^+K^0, \tag{A11b}$$

$$pd\nearrow\Lambda\Lambda nK^+K^+ \tag{A12a}$$

$$pd\searrow\Lambda\Lambda pK^+K^0. \tag{A12b}$$

Using notations analogous to those related to the discussion of reactions (A7), we find for the contributions of the $I=0$ and $I=1$ channels the following results in the case of (A11):

$$w_0=w(\mathbf{k}'+,\ \mathbf{k}''0)+w(\mathbf{k}''+,\ \mathbf{k}'0)-w(\mathbf{k}'+,\ \mathbf{k}''+)$$
$$w_1=2w(\mathbf{k}'+,\ \mathbf{k}''+)\,;$$

and in the case of (A12):

$$w_0=w(\mathbf{k}'+,\ \mathbf{k}'0)+w(\mathbf{k}''+,\ \mathbf{k}'0)-\tfrac{1}{2}w(\mathbf{k}'+,\ \mathbf{k}''+)$$
$$w_1=\tfrac{3}{2}w(\mathbf{k}'+,\ \mathbf{k}''+)\,.$$

For Bose kaons, w_0/w_1 should go to 0 like $(m_{KK}-2m_K)$ when the latter quantity goes to 0.